Communications in Applied and Industrial Mathematics, DOI: 10.1685/journal.caim.000386
ISSN 2038-0909, e-386

3D reassessment of the classical Garvin's problem

Angela Ricciardello[1], Arrigo Caserta[2], Luigia Puccio[1]

[1] *Department of Mathematics, University of Messina,*
Viale Ferdinando Stagno d'Alcontres, 31 - 98166 Messina, Italy
{aricciardello,gina}@unime.it

[2] *Istituto Nazionale di Geofisica e Vulcanologia*
Via di Vigna Murata 605 - 00143 Roma, Italy
caserta@ingv.it

Communicated by Nicola Bellomo

Abstract

The exact analytical solution for surface displacements due to a volumetric linesource disturbance buried in an elastic homogeneous and isotropic half-space is arguably due to W. W. Garvin. In this work we present a generalization in the three-dimensional space of the algorithm of Garvin's problem presented by Sanchez-Sesma and Viveros in 2006 for a point source of compressional elastic waves. Our solution can be used as a practical tool to test 3D numerical techniques devoted to simulation of the interaction dynamics between seismic radiation and near-surface geological structures. Such numerical techniques have shown to be well suited for assessing strong ground motion features. These latter represent one of the first useful information aimed at assessing the damage scenarios due to an earthquake in urban areas. Some examples are presented, and synthetic seismograms and polarigrams are displayed.

Keywords: Garvin's problem, seismic wave surface, Laplace transform.

AMS Subject Classification: 44A10, 35Q86, 86A15.

1. Introduction.

The analysis of problems concerning environmental geophysics, seismic exploration, foundation engineering, earthquake seismology and non-destructive testing of materials require the use of full-wave three-dimensional modeling methods. The wave modeling is a valuable tool for seismic interpretation and an essential part of inversion algorithm. In particular, it is important to model the surface waves, such as *Rayleigh* and *Love* waves, and record the three components of the wave field.

Rayleigh surface waves are strongly excited by a source at the free sur-

Received 2011 05 01. Accepted 2011 10 04. Published 2011 11 18.

face of a half-space. In 1904, H. Lamb gave the exact calculation in the time domain of waves generated by an impulsive line source or point source to the free surface of a half-space. Since [7], many works deal with the research of the analytical solution for such kinds of problems, like [2,5], while an elegant way of computing body wave seismograms is presented in [1,3,4]. In these works, the Cagniard-de Hoop method is applied because the Cagniard path is known analytically for a source at the free surface.

In section 2, we present the Cagniard's method applied to wave equation in a three–dimensional space. We employed Cagniard's method, as in [6,9], in which the Laplace transform is applied to transfer the Garvin's solution into the wavenumber domain. The main idea of this method can be summarized as follows. Let

$$\overline{u}(p) = \int_0^\infty g(\nu) \exp^{-pf(\nu)} d\nu, \tag{1}$$

if $f(\nu) = t$ and the inverse $\nu = f^{-1}(t) = \nu(t)$ exists, then we can write

$$\overline{u}(p) = \int_a^\infty g(\nu) \exp^{-pt} \frac{d\nu}{dt} dt = \mathcal{L}\left\{ g(t) \frac{d\nu}{dt} \right\}, \tag{2}$$

where a is a positive real number and $\overline{u}(p)$ is the Laplace transform of $u(t) = g(t)d\nu(t)/dt$. Thus,

$$u(t) = \mathcal{L}^{-1}\left\{ \overline{u}(p) \right\} = g(t) \frac{d\nu(t)}{dt} H\left(t - a\right), \tag{3}$$

where $H(\cdot)$ is the *Heaviside* function.

In paragraph 2.1, the Cagniard's path, that represents the integration domain in wavenumber space, is drawn.

Section 3 deals with the numerical results. We produce an example showing the three components of the displacement corresponding to a triangular time pulse.

Technical tools of algebraic calculations are reported in Appendix A.

Finally it is worth mentioning that the aim of this paper consists in providing a tool able to check the reliability of all numerical techniques concerning the simulation of the propagation of seismic waves, focusing on surface waves.

2. The theory.

Let us consider an elastic medium which occupies the half homogeneous medium $-\infty \leq x \leq +\infty$, $-\infty \leq y \leq +\infty$ and $z \geq 0$ and the source is a line located at $(0, 0, h)$, as shown in Figure 1.

DOI: 10.1685/journal.caim.000386

We aim to study the disturbance at a point on the surface due to a cylindrical pulse emitted from the line source.
Let $\mathbf{u} = (u_x, u_y, u_z)$ be the displacement of the point (x, y, z), obtained as solution of the wave equation

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \mu \nabla^2 \mathbf{u} + (\lambda + \mu) \nabla (\nabla \cdot \mathbf{u}) . \tag{4}$$

Then, if we introduce the potentials Φ and $\mathbf{\Psi}$ such that,

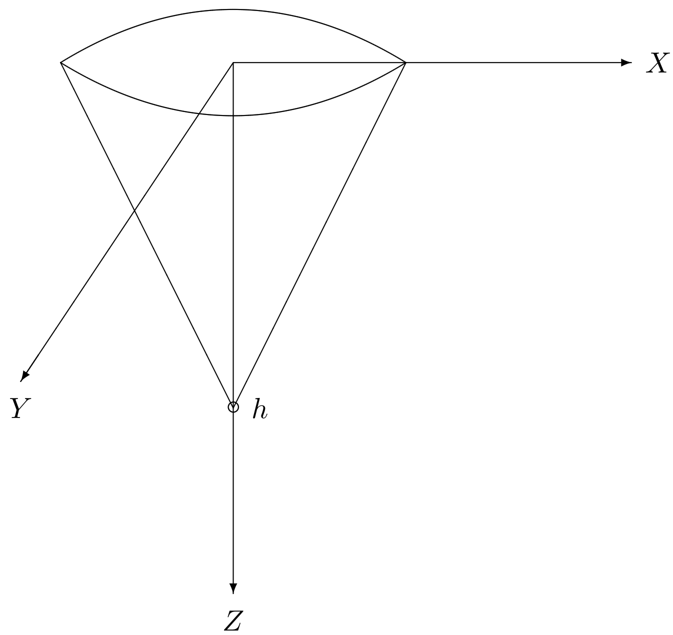


Figure 1. The line source is located on the z axis at a distance h from the free surface. The elastic medium occupies the region $-\infty < x < +\infty$, $-\infty < y < +\infty$ and $z \geq 0$.

$$\frac{\partial^2 \Phi}{\partial t^2} = \alpha \nabla^2 \Phi, \quad \frac{\partial^2 \mathbf{\Psi}}{\partial t^2} = \beta \nabla^2 \mathbf{\Psi}, \tag{5}$$

where $\alpha^2 = (\lambda + 2\mu)/\rho$ and $\beta^2 = \mu/\rho$, the displacement results

$$\mathbf{u} = -\nabla \Phi + \nabla \times \mathbf{\Psi}, \ \nabla \mathbf{\Psi} = 0. \tag{6}$$

We apply the **Laplace transforms** to (5) and we use the parameters k_1 and k_2, respectively *x-wave number* and *y-wave number*. Let $n_p = \sqrt{k_1^2 + k_2^2 + \frac{p^2}{\alpha^2}}$ and $n_s = \sqrt{k_1^2 + k_2^2 + \frac{p^2}{\beta^2}}$, we obtain

- $\overline{\Phi} = \overline{\Phi}_0 + \int_0^{+\infty} \int_0^{+\infty} B(k_1, k_2) \exp^{-zn_p} \cos(k_1 x) \cos(k_2 y) dk_1 dk_2,$

  where $\overline{\Phi}_0$, that is the solution in the absence of the free boundary at $z = 0$, is given by

  $$\overline{\Phi}_0 = \int_0^{+\infty} \int_0^{+\infty} \frac{\overline{f}(p)}{n_p} \exp^{-(h-z)n_p} \cos(k_1 x) \cos(k_2 y) dk_1 dk_2, \tag{7}$$

  as shown in [8]. So we have

  $$\overline{\Phi} = \int_0^{+\infty} \int_0^{+\infty} \left(\frac{\overline{f}(p)}{n_p} \exp^{-(h-z)n_p} + B(k_1, k_2) \exp^{-zn_p} \right) \cos(k_1 x) \cos(k_2 y) dk_1 dk_2;$$

- $\overline{\Psi}(1) = \int_0^{+\infty} \int_0^{+\infty} C(k_1, k_2) \exp^{-zn_s} \cos(k_1 x) \sin(k_2 y) dk_1 dk_2;$
- $\overline{\Psi}(2) = \int_0^{+\infty} \int_0^{+\infty} D(k_1, k_2) \exp^{-zn_s} \sin(k_1 x) \cos(k_2 y) dk_1 dk_2;$
- $\overline{\Psi}(3) = \int_0^{+\infty} \int_0^{+\infty} E(k_1, k_2) \exp^{-zn_s} \sin(k_1 x) \sin(k_2 y) dk_1 dk_2.$

Note that $\overline{u}_1(x, y, z)$ is an odd function of x and an even function of y; $\overline{u}_2(x, y, z)$ is an even function of x and an odd function of y; $\overline{u}_3(x, y, z)$ is an even function of x and an even function of y.

Applying the null shear condition, $\nabla\Psi = 0$, we obtain

$$E(k_1, k_2) = -\frac{1}{n_s} \left(k_1 C(k_1, k_2) + k_2 D(k_1, k_2) \right).$$

By condition (6), we have

$$\begin{aligned} \overline{u}_1(x, y, z) &= -\frac{\partial \overline{\Phi}}{\partial x} + \left(\frac{\partial \overline{\Psi}(3)}{\partial y} - \frac{\partial \overline{\Psi}(2)}{\partial z} \right) \\ &= \int_0^{+\infty} \int_0^{+\infty} \Big[\, k_1 B(k_1, k_2) \exp^{-zn_p} \\ &\quad + \left(-\frac{k_1 k_2}{n_s} C(k_1, k_2) + (n_s - \frac{k_2^2}{n_s}) D(k_1, k_2) \right) \exp^{-zn_s} \\ &\quad + k_1 \frac{\overline{f}(p)}{n_p} \exp^{-(h-z)n_p} \Big] \sin(k_1 x) \cos(k_2 y) dk_1 dk_2; \end{aligned}$$

$$
\begin{aligned}
\overline{u}_2(x,y,z) =& -\frac{\partial\overline{\Phi}}{\partial y} + \left(\frac{\partial\overline{\Psi}(1)}{\partial z} - \frac{\partial\overline{\Psi}(3)}{\partial x}\right) \\
=& \int_0^{+\infty}\int_0^{+\infty} \Big[k_2 B(k_1,k_2)\exp^{-zn_p} \\
&+ \left(-(n_s - \frac{k_1^2}{n_s})C(k_1,k_2) + \frac{k_1k_2}{n_s}D(k_1,k_2)\right)\exp^{-zn_s} \\
&+ \frac{k_2}{n_p}\overline{f}(p)\exp^{-(h-z)n_p}\Big]\cos(k_1x)\sin(k_2y)dk_1dk_2;
\end{aligned}
$$

$$
\begin{aligned}
\overline{u}_3(x,y,z) =& -\frac{\partial\overline{\Phi}}{\partial z} + \left(\frac{\partial\overline{\Psi}(1)}{\partial y} - \frac{\partial\overline{\Psi}(2)}{\partial x}\right) \\
=& \int_0^{+\infty}\int_0^{+\infty} \Big[\Big(-\overline{f}(p)\exp^{-(h-z)n_p} + n_pB(k_1,k_2)\exp^{-zn_p}\Big) \\
&+ (k_2C(k_1,k_2) - k_1D(k_1,k_2))\exp^{-zn_s}\Big] \\
&\cdot\cos(k_1x)\cos(k_2y)dk_1dk_2.
\end{aligned}
$$

In order to find suitable functions $B(k_1,k_2)$, $C(k_1,k_2)$ and $D(k_1,k_2)$, we impose that, at the surface $z=0$, the normal stress $\overline{\sigma}_{zz}$, $\overline{\sigma}_{xz}$ and $\overline{\sigma}_{yz}$ must vanish. Solving the linear system so defined, (see the Appendix), we have:

$$
\begin{aligned}
\overline{u}_1(x,y,0) =& \Re\int_0^{\infty}\Im\int_0^{\infty}\Big\{\frac{1}{\Delta}\Big[-4n_pn_s\left(p^2+(\alpha^2-\beta^2)(k_1^2+k_2^2)\right) \\
&+\frac{p^2}{\beta^2}\left(2\beta^2n_s^2-p^2\right)\Big]+1\Big\} \\
&\cdot\frac{k_1}{n_p}\overline{f}(p)\exp^{-hn_p-i(k_1x+k_2y)}\,dk_1dk_2;
\end{aligned}
$$

$$
\begin{aligned}
\overline{u}_2(x,y,0) =& \Im\int_0^{\infty}\Re\int_0^{\infty}\Big\{\frac{1}{\Delta}\Big[-4n_pn_s\left(p^2+(\alpha^2-\beta^2)(k_1^2+k_2^2)\right) \\
&+\frac{p^2}{\beta^2}\left(2\beta^2n_s^2-p^2\right)\Big]+1\Big\} \\
&\cdot\frac{k_2}{n_p}\overline{f}(p)\exp^{-hn_p-i(k_1x+k_2y)}\,dk_1dk_2;
\end{aligned}
$$

$$\begin{aligned}\overline{u}_3(x,y,0) =& \Re\int_0^{\infty}\Re\int_0^{\infty}\left\{-1+\frac{1}{\Delta}\left[\frac{p^2}{\beta^2}\left(2\beta^2 n_s^2-p^2\right)\right.\right.\\ &-4n_p n_s(\alpha^2-\beta^2)(k_1^2+k_2^2)\\ &\left.\left.+4(2\beta^2 n_s^2-p^2)(k_1^2+k_2^2)\right]\right\}\\ &\cdot\overline{f}(p)\exp^{-hn_p-i(k_1x+k_2y)}\,dk_1dk_2.\end{aligned}$$

Thus, we introduce the cylindric coordinate (ω, q)

$$k_1 = p\left(\omega\cos(\phi) - q\sin(\phi)\right), \qquad k_2 = p\left(\omega\sin(\phi) + q\cos(\phi)\right);$$
$$x = r\cos(\phi), \qquad y = r\sin(\phi).$$

For the sake of simplicity, we choose $\overline{f}(p) = a/2p^2$ that is the Laplace transform of the function $f(t) = (at)/2$. Let $s = i\omega$, then, according to calculations reported in the Appendix, relating to Equation (17), we have the following formulas
(8)

$$\overline{u}_1(r,\phi,0) = p^2\overline{f}(p)\Re\int_0^{+\infty}\Im\int_0^{+\infty}F_1(s,q)\exp^{-p(h\sqrt{\omega^2+q^2+\alpha^{-2}}-i\omega r)}\,d\omega dq;$$

$$\overline{u}_2(r,\phi,0) = p^2\overline{f}(p)\Im\int_0^{+\infty}\Re\int_0^{+\infty}F_2(s,q)\exp^{-p(h\sqrt{\omega^2+q^2+\alpha^{-2}}-i\omega r)}\,d\omega dq;$$

$$\overline{u}_3(r,\phi,0) = p^2\overline{f}(p)\Re\int_0^{+\infty}\Re\int_0^{+\infty}F_3(s,q)\exp^{-p(h\sqrt{\omega^2+q^2+\alpha^{-2}}-i\omega r)}\,d\omega dq;$$

$$\begin{aligned}\Delta^{\star\star} = \Big[&4\sqrt{q^2-s^2+\beta^{-2}}\sqrt{q^2-s^2+\alpha^{-2}}\left(\alpha^2-\beta^2\right)\left(q^2-s^2\right)\\ &-2\left(q^2-s^2+\beta^{-2}\right)-\beta^{-2}\Big]\end{aligned}$$

and

$$\begin{aligned}F_1(s,q) =& 1-\frac{1}{\Delta^{\star\star}}\Big[4\sqrt{q^2-s^2+\beta^{-2}}\sqrt{q^2-s^2+\alpha^{-2}}\\ &\cdot\left(1+\left(\alpha^2-\beta^2\right)\left(q^2-s^2\right)\right)\\ &+2\left(q^2-s^2+\alpha^{-2}\right)-\beta^{-2}\Big]\\ &\cdot\frac{q\sin(\Phi)+i\ s\cos(\Phi)}{\sqrt{q^2-s^2+\alpha^{-2}}};\end{aligned}$$

DOI: 10.1685/journal.caim.000386

$$
\begin{aligned}
F_2(s,q) =& 1 + \frac{1}{\Delta^{\star\star}} \Big[4\sqrt{q^2 - s^2 + \beta^{-2}} \sqrt{q^2 - s^2 + \alpha^{-2}} \\
& \cdot \left(1 + \left(\alpha^2 - \beta^2\right)\left(q^2 - s^2\right)\right) \\
& + 2\left(q^2 - s^2 + \alpha^{-2}\right) - \beta^{-2} \Big] \\
& \cdot \frac{q\cos(\Phi) - i\ s\sin(\Phi)}{\sqrt{\omega^2 + q^2 + \alpha^{-2}}};
\end{aligned}
$$

$$
\begin{aligned}
F_3(s,q) =& 1 - \frac{1}{\Delta^{\star\star}} \Big[4\sqrt{q^2 - s^2 + \beta^{-2}} \sqrt{q^2 - s^2 + \alpha^{-2}} \\
& \cdot \left(\alpha^2 - \beta^2\right)\left(q^2 - s^2\right) \\
& + 2\left(q^2 - s^2 + \alpha^{-2}\right) - \beta^{-2} \\
& - 8\beta^2 (q^2 - s^2 + \beta^{-2}(q^2 - s^2)) \Big] \\
& \cdot \frac{q\sin(\Phi) + i\ s\cos(\Phi)}{\sqrt{\omega^2 + q^2 + \alpha^{-2}}}.
\end{aligned}
$$

2.1. *The Cagniard path.*

The main feature of Cagniard-de Hoop method for line source problems, in a two-dimensional space, is that it may allow to find the exact algebraic solution. On the contrary, for three-dimensional problems, the solution is generally expressed by three integrals, one for each component, that are evaluated over a finite segment of the Cagniard path, as shown in Figure 2. Let

$$
t = -h\sqrt{q^2 + \alpha^{-2} - s^2} + sr, \tag{9}
$$

whose inverse transformation is

$$
s = \begin{cases} \dfrac{rt - h\sqrt{R^2\left(\alpha^{-2} + q^2\right) - t^2}}{R^2} & if \quad t \leq \sqrt{R^2\left(\alpha^{-2} + q^2\right)} \\[2ex] \dfrac{rt + ih\sqrt{t^2 - R^2\left(\alpha^{-2} + q^2\right)}}{R^2} & if \quad t \geq \sqrt{R^2\left(\alpha^{-2} + q^2\right)} \end{cases} \tag{10}
$$

with $R^2 = r^2 + (h - z)^2$.
We obtain the following points

$$\begin{cases} t = 0 \\ s = -\dfrac{h}{R}\sqrt{\dfrac{1}{\alpha^2} + q^2} \end{cases}$$

$$\begin{cases} t = R\sqrt{\dfrac{1}{\alpha^2} + q^2} \\ s = \dfrac{r}{R}\sqrt{\dfrac{1}{\alpha^2} + q^2} \end{cases}$$

where

$$R = \sqrt{r^2 + h^2}.$$

In Figure 2, note that, the positive real axis of the *s-plane* becomes asymptotic to the straight line making the angle $j = \tan^{-1}\dfrac{r}{h}$. In addition, when $t = h\sqrt{\frac{1}{\alpha^2} + q^2}$ then $s = 0$.

Moreover, in order to evaluate

$$\int_{\mathcal{C}} \exp^{-pt} \tilde{F}_i(t,q)\frac{ds}{dt}dqdt \quad \forall i = 1, 2, 3,$$

note that, by the *Cagniard Path*, the integration with respect to t is shown in Figure 3(a).
After simple algebra, it results the integration with respect to q illustrated in Figure 3(b). Let $A = \sqrt{\frac{t^2}{h^2} - \frac{1}{\alpha^2}}$, then

$$\int_0^\infty \exp^{-pt} H(t - \frac{h}{\alpha}) \left[\int_0^A \left(\tilde{F}_i(t,q)\frac{ds}{dt}\right) dq\right] dt.$$

It means that, if we apply the anti-Laplace transform, we obtain

$$u_i(r, \phi, 0) = \frac{a}{2}\left[\int_0^A \tilde{F}_i(t,q)\frac{ds}{dt}dq\right] H(t - \frac{h}{\alpha}). \tag{11}$$

The value $\displaystyle\int_0^A \tilde{F}_i(t,q)\frac{ds}{dt}dq$ is approximated by means of numerical integration, as shown in section 3.

2.2. *Initial displacement.*

Let us determine the initial displacement u_{r0}, according to the scheme presented in [2]. From (7)

DOI: 10.1685/journal.caim.000386

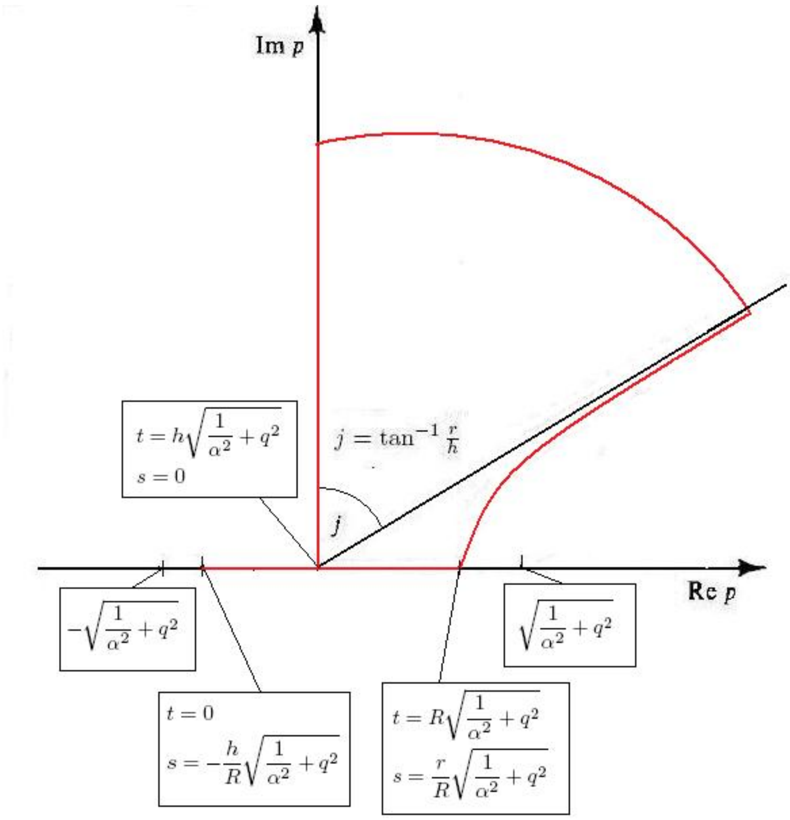
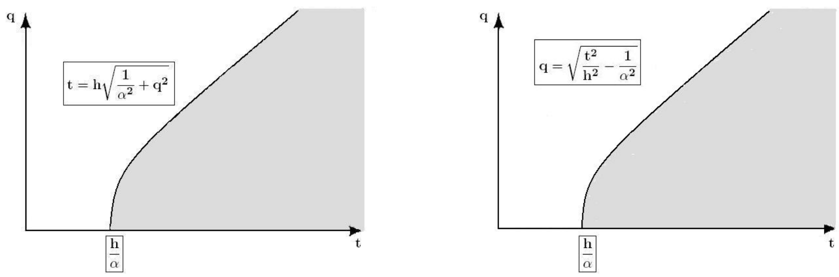


Figure 2. The Cagniard Path is represented by the red line.



(a) t-integration (b) q-integration

Figure 3. The domain of integration with respect to t and q variables is shown.

$$\overline{\Phi}_0 = \overline{f}(p)\int_0^\infty\int_0^\infty \frac{1}{n_p}\exp^{-(h-z)n_p+i(k_1x+k2y)}\,dk_1dk_2.$$

By using variables (t, q), previously introduced, it can be written as

$$\begin{aligned}\overline{\Phi}_0 =& \frac{a}{2}\int_0^\infty \int_0^\infty \frac{H\left[t - R\sqrt{\alpha^{-2}+q^2}\right]}{\sqrt{t^2 - R^2(\alpha^{-2}+q^2)}} dq \exp^{-pt} dt \\ =& \frac{a}{2}\int_0^\infty \left\{ H\left[t - R\sqrt{\alpha^{-2}+q^2}\right] \int_0^A \frac{dq}{\sqrt{t^2 - R^2(\alpha^{-2}+q^2)}} \right\} \exp^{-pt} dt.\end{aligned}$$

So

$$\begin{aligned}\Phi_0 =& \frac{a}{2} H\left[t - \frac{R}{\alpha}\right] \int_0^A \frac{dq}{\sqrt{t^2 - R^2(\alpha^{-2}+q^2)}} \\ =& \frac{a\pi H\left[t - \frac{R}{\alpha}\right]}{4R} = \frac{a\pi H\left[t - \frac{R}{\alpha}\right]}{4\sqrt{r^2+h^2}}.\end{aligned}$$

Then, physical displacement is

$$u_{r0} = -\frac{\partial}{\partial r}\Phi_0 = -\frac{\partial}{\partial r}\frac{a\pi H\left[t - \frac{R}{\alpha}\right]}{4\sqrt{r^2+h^2}} = \frac{ar\pi H\left[t - \frac{R}{\alpha}\right]}{4(r^2+h^2)^{\frac{3}{2}}}.$$

3. Numerical results.

In this section we present the numerical integration of Equation (11). Instead of having a step function as a time signal, we consider a source time signal corresponding to a ramp, as follows:

$$\langle t \rangle = \begin{cases} 0, & t < 0 \\ t, & t \geq 0. \end{cases} \tag{12}$$

Therefore, we can consider the following triangular load of duration d centered at $t = d/2$,

$$\langle t \rangle - 2\langle t - d/2 \rangle + \langle t - d \rangle . \tag{13}$$

Note that once the displacement corresponding to a triangular load is obtained, the displacement for almost any given time signal can be computed by convolution of the signal with the triangular load using fast Fourier transform.

In order to integrate numerically Equation (11), a second order Simpson rule was used.

Twelve equally spaced collocation points are located on the surface $z = 0$, with a distance of 5 km. The buried source line is at $h = 1$ km. The wave velocities are $\alpha = 1.732$ km/sec and $\beta = 1$ km/sec. The duration of the source is $d = 3$ sec and the maximum amplitude occurs at $d/2$. The direct P wave reaches first the target, while the Rayleigh wave arrives later, because the Rayleigh waves have a speed less than P-waves. Moreover, Rayleigh waves are surface waves, in the sense that they travel across the surface and they decrease exponentially in amplitude as the distance from the surface increases.

Since $\phi = 0$ has been chosen, the polarization may be visualized in the plane xz. For this reason, the Rayleigh waves are depicted in Figure 4 and Figure 6. In particular, their amplitude is greater for the vertical displacement u_z than for the horizontal one u_x. Otherwise, the Rayleigh waves can not be displayed in the u_y displacement shown in Figure 5.

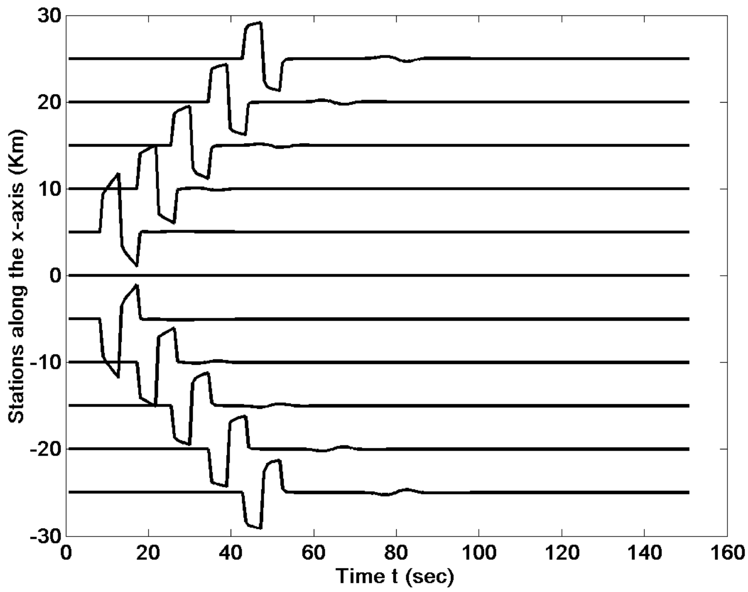


Figure 4. The displacement u_x. The first arrive represents the P-wave, while the second one is the slower Rayleigh one.

4. Conclusions.

F. Sanchez-Sesma et al, in [9], extended the classic Garvin's solution, for virtually any given time signal, turning it into a useful tool to assess results obtained with numerical methods, such as finite-difference, pseudo-spectral or finite element method, in a bi-dimensional space. In this work,

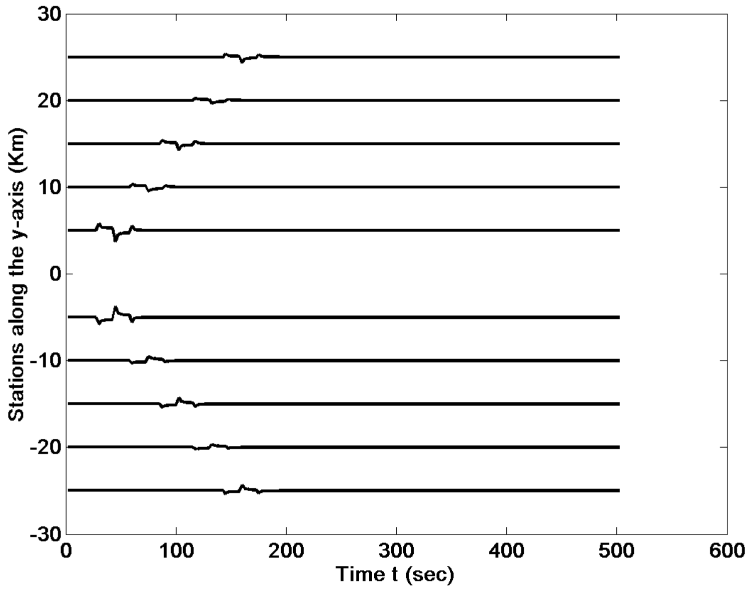


Figure 5. The displacement u_y. The P-wave is only depicted because the polarization plane is xz, so the surface waves can not be registered along its orthogonal direction.

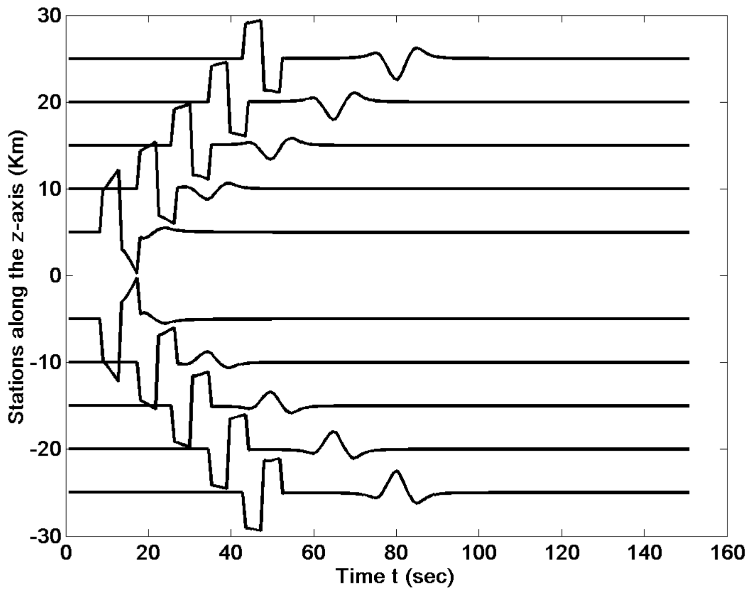


Figure 6. The displacement u_z. The first arrive represents the P-wave, while the second one is the slower Rayleigh wave.

DOI: 10.1685/journal.caim.000386

we presented its generalization in a three-dimensional space. All details about developed calculations are reported in the Appendix.

This work provides a tool which is a great help for testing numerical 3D codes devoted to simulate the dynamics of interaction between seismic radiation and near surface geology structures. One of the main applications of such studies is in seismic risk mitigation, with special attention to urban areas.

Acknowledgments.

This work has been supported by GNCS-INdAM, and Research Grants of the University of Messina. The authors wish to thank the anonymous Referee, whose comments were useful to improve the paper.

REFERENCES

1. I.D. Achenbach, *Wave propagation in elastic solids. North Holland Puhl. Co.*, (1975).
2. K. Aki, P.G. Richards, *Quantitative Seismology. WH Freeman and Co.*, 1, (1980).
3. L. Cagniard, *Reflection et Refraction des Ondes Seismique Prograssives. Gauthiers-Villars, Paris*, (1939).
4. A.T. de Hoop, *A modification of Cagniards method for solvingseismic pulse problems. Appl. Sci. Res.*, 8, (1960), 349–356.
5. W.M. Ewing, W.S. Jardetzky, F. Press, *Elastic waves in layered media. McGraw-Hill Book Co.*, (1957).
6. W.W. Garvin, *Exact transient solution of the buried line source problem. Proc. R. Soc. London Ser. A*, 234 (1956) 528–541.
7. H. Lamb, *On the propagation of tremors over the surface of an elastic solid. Philos. Tran. R. Soc. London Ser. A*, 203 (1904) 142.
8. E.R. Lapwood, *The disturbance due to a line source in a semiinfinite elastic medium. Philos. Trans. R. Soc. London* Ser. A 841, (1949), 63–100.
9. F. Sanches-Sesma, U. Iturraran-Viveros, *The Classic Garvin's Problem Revisited. Bull. Seismological Soc. America*, 96 (2006) 1344–1351.

A. Appendix.

By the free boundary condition, it follows that

$$\begin{cases} \overline{\sigma}_{xz} = 0 \\ \overline{\sigma}_{yz} = 0 \\ \overline{\sigma}_{zz} = 0 \end{cases} \tag{14}$$

or rather

$$\begin{cases} \dfrac{\partial \overline{u}_1}{\partial z} + \dfrac{\partial \overline{u}_3}{\partial x} = 0 \\ \dfrac{\partial \overline{u}_2}{\partial z} + \dfrac{\partial \overline{u}_3}{\partial y} = 0 \\ (\alpha^2 - 2\beta^2)\left(\dfrac{\partial \overline{u}_1}{\partial x} + \dfrac{\partial \overline{u}_2}{\partial y}\right) + \alpha^2 \dfrac{\partial \overline{u}_3}{\partial z} = 0. \end{cases} \tag{15}$$

So we have

$$\overline{\sigma}_{xz} = \int_0^{+\infty} \int_0^{+\infty} \big\{ -2k_1 n_p B(k_1, k_2) + (k_1^2 - n_s^2 + k_2^2) D(k_1, k_2) \\ +2k_1 \overline{f}(p) \exp^{-hn_p} \Big\} \sin(k_1 x) \cos(k_2 y) dk_1 dk_2,$$

$$\overline{\sigma}_{yz} = \int_0^{+\infty} \int_0^{+\infty} \big\{ -2k_2 n_p B(k_1, k_2) - (k_2^2 + k_1^2 - n_s^2) C(k_1, k_2) \\ +2k_2 \overline{f}(p) \exp^{-hn_p} \Big\} \cdot \cos(k_1 x) \sin(k_2 y) dk_1 dk_2,$$

$$\overline{\sigma}_{zz} = \int_0^{+\infty} \int_0^{+\infty} \big\{ -\left(2\beta^2 n_s^2 - p^2\right) B(k_1, k_2) - 2\left(\alpha^2 - \beta^2\right) k_2 n_s C(k_1, k_2) \\ +2\left(\alpha^2 - \beta^2\right) k_2 n_s D(k_1, k_2) - \frac{1}{n_p}\left[2\beta^2 n_s^2 - p^2\right] \overline{f}(p) \exp^{-hn_p} \Big\} \\ \cdot \cos(k_1 x) \cos(k_2 y) dk_1 dk_2.$$

That is

$$(16)\quad \begin{cases} 2k_1 n_p B(k_1,k_2) + \dfrac{p^2}{\beta^2} D(k_1,k_2) = 2k_1 \overline{f}(p) \exp^{-hn_p} \\[2ex] 2k_2 n_p B(k_1,k_2) - \dfrac{p^2}{\beta^2} C(k_1,k_2) = 2k_2 \overline{f}(p) \exp^{-hn_p} \\[2ex] -\left(2\beta^2 n_s^2 - p^2\right) B(k_1,k_2) - 2\left(\alpha^2 - 2\beta^2\right) k_2 n_s C(k_1,k_2) + \\ + 2\left(\alpha^2 - 2\beta^2\right) k_1 n_s D(k_1,k_2) = \dfrac{1}{n_p}\left(2\beta^2 n_s^2 - p^2\right) \overline{f}(p) \exp^{-hn_p}. \end{cases}$$

We solve the system

$$\begin{vmatrix} 2n_p k_1 & 0 & \frac{p^2}{\beta^2} \\ 2n_p k_2 & -\frac{p^2}{\beta^2} & 0 \\ -2\beta^2 n_s^2 + p^2 & -2k_2 n_s(\alpha^2 - \beta^2) & -2k_1 n_s(\alpha^2 - \beta^2) \end{vmatrix}$$

$$= 4(\alpha^2 - \beta^2)\frac{p^2}{\beta^2} n_p n_s (k_1^2 + k_2^2) - \frac{p^4}{\beta^4}(2\beta^2 n_s^2 - p^2).$$

Then we obtain

$$\begin{aligned} B(k_1,k_2) =& \frac{\overline{f}(p)}{\Delta}\left[\frac{1}{n_p}\frac{p^2}{\beta^2}\left(2\beta^2 n_s^2 - p^2\right) - 4n_s(\alpha^2 - \beta^2)(k_1^2 + k_2^2)\right] \exp^{-hn_p}, \\ C(k_1,k_2) =& 4k_2 \frac{\overline{f}(p)}{\Delta}\left(2\beta^2 n_s^2 - p^2\right) \exp^{-hn_p}, \\ D(k_1,k_2) =& -4k_2 \frac{\overline{f}(p)}{\Delta}\left(2\beta^2 n_s^2 - p^2\right) \exp^{-hn_p}, \end{aligned}$$

where $\Delta = 4\left(\alpha^2 - \beta^2\right) n_p n_s \left(k_1^2 + k_2^2\right) - \frac{p^2}{\beta^2}\left(2\beta^2 n_s^2 - p^2\right)$.

Substituting

$$\overline{u}_1(x,y,z) = \int_0^\infty \int_0^\infty \left\{ \frac{k_1}{n_p} \exp^{-(h-z)n_p} + \frac{k_1}{\Delta} \exp^{-(h+z)n_p} \right.$$
$$\cdot \left[-4n_s \left(\alpha^2 - \beta^2\right) \left(k_1^2 + k_2^2\right) \right.$$
$$\left. \left. + \frac{1}{n_p} \left(2\beta^2 n_s^2 - p^2\right) \right] - \frac{4k_1 n_s p^2}{\Delta} \exp^{-hn_p - zn_s} \right\}$$
$$\cdot \overline{f}(p) \sin(k_1 x) \cos(k_2 y) dk_1 dk_2,$$

$$\overline{u}_2(x,y,z) = \int_0^\infty \int_0^\infty \left\{ \frac{k_2}{n_p} \exp^{-(h-z)n_p} + \frac{k_2}{\Delta} \exp^{-(h+z)n_p} \right.$$
$$\cdot \left[-4n_s \left(\alpha^2 - \beta^2\right) \left(k_1^2 + k_2^2\right) \right.$$
$$\left. \left. + \frac{1}{n_p} \left(2\beta^2 n_s^2 - p^2\right) \right] - \frac{4k_2 n_s p^2}{\Delta} \exp^{-hn_p - zn_s} \right\}$$
$$\cdot \overline{f}(p) \cos(k_1 x) \sin(k_2 y) dk_1 dk_2,$$

$$\overline{u}_3(x,y,z) = \int_0^\infty \int_0^\infty \left\{ \exp^{-(h-z)n_p} + \frac{n_p}{\Delta} \exp^{-hn_p - zn_s} \right.$$
$$\cdot \left[-4n_s \left(\alpha^2 - \beta^2\right) \left(k_1^2 + k_2^2\right) \right.$$
$$\left. + \frac{1}{n_p} \left(2\beta^2 n_s^2 - p^2\right) \right]$$
$$\left. + 4 \left(2\beta^2 n_s^2 - p^2\right) \left(k_1^2 + k_2^2\right) \frac{\exp^{-hn_p - zn_s}}{\Delta} \right\}$$
$$\cdot \overline{f}(p) \cos(k_1 x) \cos(k_2 y) dk_1 dk_2.$$

If we consider $z = 0$, we have

$$\overline{u}_1(x,y,0) = \int_0^\infty \int_0^\infty \left\{ 1 + \frac{1}{\Delta} \left[-4n_p n_s \left(p^2 + (\alpha^2 - \beta^2)(k_1^2 + k_2^2) \right) \right] + \frac{p^2}{\beta^2} \left(2\beta^2 n_s^2 - p^2 \right) \right\} \frac{k_1}{n_p} \overline{f}(p) \exp^{-hn_p} \sin(k_1 x) \cos(k_2 y) dk_1 dk_2,$$

$$\overline{u}_2(x,y,0) = \int_0^\infty \int_0^\infty \left\{ 1 + \frac{1}{\Delta} \left[-4n_p n_s \left(p^2 + (\alpha^2 - \beta^2)(k_1^2 + k_2^2) \right) \right] + \frac{p^2}{\beta^2} \left(2\beta^2 n_s^2 - p^2 \right) \right\} \frac{k_2}{n_p} \overline{f}(p) \exp^{-hn_p} \cos(k_1 x) \sin(k_2 y) dk_1 dk_2,$$

$$\overline{u}_3(x,y,0) = \int_0^\infty \int_0^\infty \left\{ 1 + \frac{1}{\Delta} \left[-4n_p n_s (\alpha^2 - \beta^2)(k_1^2 + k_2^2) + \frac{p^2}{\beta^2} \left(2\beta^2 n_s^2 - p^2 \right) + 4(2\beta^2 n_s^2 - p^2)(k_1^2 + k_2^2) \right\} \cdot \overline{f}(p) \exp^{-hn_p} \cos(k_1 x) \cos(k_2 y) dk_1 dk_2.$$

Using the cylindrical coordinate (ω, q), it follows

$$dk_1 dk_2 = p^2 d\omega dq,$$

$$n_p = p\sqrt{\omega^2 + q_2 + \alpha^{-2}}$$

and

$$n_s = p\sqrt{\omega^2 + q^2 + \beta^{-2}}.$$

Substituting, we obtain

$$\overline{u}_1(r,\phi,0) = p^2 \overline{f}(p) \Re \int_0^{+\infty} \Im \int_0^{+\infty} F_1(\omega,q) \exp^{-p(h\sqrt{\omega^2+q^2+\alpha^{-2}} - i\omega r)} d\omega dq,$$

$$\overline{u}_2(r,\phi,0) = p^2 \overline{f}(p) \Im \int_0^{+\infty} \Re \int_0^{+\infty} F_2(\omega,q) \exp^{-p(h\sqrt{\omega^2+q^2+\alpha^{-2}} - i\omega r)} d\omega dq, \quad (17)$$

$$\overline{u}_3(r,\phi,0) = p^2 \overline{f}(p) \Re \int_0^{+\infty} \Re \int_0^{+\infty} F_3(\omega,q) \exp^{-p(h\sqrt{\omega^2+q^2+\alpha^{-2}} - i\omega r)} d\omega dq,$$

where

$$F_1(r,\phi) = 1 + \frac{1}{\Delta^\star}\Big[4\sqrt{\omega^2+q^2+\beta^{-2}}\sqrt{\omega^2+q^2+\alpha^{-2}}\left(1+\left(\alpha^2-\beta^2\right)\left(\omega^2+q^2\right)\right)$$
$$+2\left(\omega^2+q^2+\alpha^{-2}\right)-\beta^{-2}\Big]\frac{\omega\cos(\Phi)-q\sin(\Phi)}{\sqrt{\omega^2+q^2+\alpha^{-2}}},$$

$$F_2(r,\phi) = 1 + \frac{1}{\Delta^\star}\Big[4\sqrt{\omega^2+q^2+\beta^{-2}}\sqrt{\omega^2+q^2+\alpha^{-2}}\left(1+\left(\alpha^2-\beta^2\right)\left(\omega^2+q^2\right)\right)$$
$$+2\left(\omega^2+q^2+\alpha^{-2}\right)-\beta^{-2}\Big]\frac{\omega\sin(\Phi)+q\cos(\Phi)}{\sqrt{\omega^2+q^2+\alpha^{-2}}},$$

$$F_3(r,\phi) = 1 + \frac{1}{\Delta^\star}\Big[4\sqrt{\omega^2+q^2+\beta^{-2}}\sqrt{\omega^2+q^2+\alpha^{-2}}\left(\alpha^2-\beta^2\right)\left(\omega^2+q^2\right)$$
$$+2\left(\omega^2+q^2+\alpha^{-2}\right)-\beta^{-2}-8\beta^2(\omega^2+q^2+\beta^{-2}(\omega^2+q^2))\Big]$$
$$\frac{\omega\cos(\Phi)-q\sin(\Phi)}{\sqrt{\omega^2+q^2+\alpha^{-2}}}$$

and

$$\Delta^\star = \Big[4\sqrt{\omega^2+q^2+\beta^{-2}}\sqrt{\omega^2+q^2+\alpha^{-2}}\left(\alpha^2-\beta^2\right)\left(\omega^2+q^2\right)-$$
$$-2\left(\omega^2+q^2+\beta^{-2}\right)-\beta^{-2}\Big].$$