

On a class of Hamilton-Jacobi equations with related Logarithmic Sobolev Inequality, and optimality

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Abstract

In this paper, we establish a new optimal logarithmic Sobolev type inequality. The proof is based on the analysis of the Cauchy problem for a first order Hamilton-Jacobi equation and on qualitative properties of solutions.

Keywords: Hamilton-Jacobi equations, Hypercontractivity, Hopf-Lax formula, Logarithmic Sobolev Inequalities.

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1. Introduction.

The classical logarithmic Sobolev inequality, proved by Gross (1975, see [1]) may be written as

$$(1) \quad \mathcal{E}_\mu(u^2) \leq \frac{2}{\rho} \int_{\mathbb{R}^N} |Du|^2 d\mu,$$

where μ is a probability measure, ρ is a positive real number, u is a smooth enough function defined in \mathbb{R}^N and \mathcal{E}_μ is the entropy, defined by

$$(2) \quad \mathcal{E}_\mu(u^2) = \int_{\mathbb{R}^N} u^2 \log u^2 d\mu - \int_{\mathbb{R}^N} u^2 d\mu \log \int_{\mathbb{R}^N} u^2 d\mu.$$

In the paper [1] the equivalence between (1) and the hypercontractivity of the linear semigroup associated to the heat equation has been proved. The various fields of application and the different shapes of the Logarithmic Sobolev Inequalities (LSI, in the following) are too many to be cited here. We just mention [2], where LSI for numerical schemes in dimension

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one and their application to Monte Carlo and ergodic simulations are described, and [3], where the authors have highlighted tight links among LSI, certain concentration inequalities due to Talagrand and diffusion equations. Furthermore, we suggest papers [4] and [5] to the interested reader, in order to get some useful informations on the subject. Related to this paper, we refer to the articles by Bobkov, Gentil, and Ledoux [5] and by Gentil [6]. In the papers [5] and [6] it is proved that LSI are related to hypercontractivity of the nonlinear semigroups, leading to Hopf-Lax type formulas and a class of Hamilton-Jacobi equations. More precisely, in [5] and [6] the initial value problem

$$(3) \quad \begin{cases} u_t + \frac{1}{2} |Du|^2 = 0 & \text{in } \mathbb{R}^N \times \mathbb{R}^+ \\ u = u_0 & \text{in } \mathbb{R}^N \times \{t = 0\} \end{cases},$$

is considered. The solutions are described by the Hopf-Lax formula

$$Q_t^* u_0(x) = \inf_{y \in \mathbb{R}^N} \left[u_0(y) + \frac{1}{2t} |x - y|^2 \right], \quad x \in \mathbb{R}^N, \quad t > 0.$$

Under suitable hypothesis on u_0 (see [7] for more details), the semigroup $(Q_t^*)_{t>0}$ defines the unique viscosity solution $u(x, t) = Q_t^*(u_0)$ of (3). By means of the derivative of the norm function $F(t) = \|e^{Q_t^*(u_0)}\|_{L^{q(t)}}$, with $q(t)$ a non decreasing C^1 function, which verifies the conditions $q(0) = p$ and $q'(t) = \rho$, it is easy to obtain that (1) implies the hypercontractivity inequality

$$(4) \quad \left\| e^{Q_t^* u_0} \right\|_{L^{p+\rho t}(\mathbb{R}^N)} \leq \|e^{u_0}\|_{L^p(\mathbb{R}^N)}, \quad \forall t > 0.$$

Vice versa, if (4) holds for all $t > 0$ and some $p \neq 0$, then (1) is verified. In the spirit of works [5] and [6], the problem

$$(5) \quad \begin{cases} u_t + \frac{1}{2} |Du|^2 + \sum_{i=1}^N \alpha_i x_i u_{x_i} = 0 & \text{in } \mathbb{R}^N \times \mathbb{R}^+ \\ u = u_0 & \text{in } \mathbb{R}^N \times \{t = 0\} \end{cases},$$

where $\alpha_1, \dots, \alpha_N$ are positive real numbers, is studied in the first part of the paper [8]. There it is proved the hypercontractivity of the nonlinear semigroup Q_t^α related to problem (5), its optimality and stated the LSI

$$(6) \quad \mathcal{E}(u_0^2) + N \int_{\mathbb{R}^N} u_0^2(x) dx \leq \frac{2}{\pi} \int_{\mathbb{R}^N} |Du_0(x)|^2 dx,$$

for any admissible function u_0 , with

$$(7) \quad \mathcal{E}(w) = \int_{\mathbb{R}^N} w \log w dx - \int_{\mathbb{R}^N} w dx \log \int_{\mathbb{R}^N} w dx.$$

In the second part of [8], the mixed Cauchy problem

$$(8) \quad \begin{cases} u_t(x, x', t) + \frac{1}{2} |Du(x, x', t)|^2 + \frac{1}{2} |D'u(x, x', t)|^2 + \\ + \sum_{i=1}^n \alpha_i x_i u_{x_i}(x, x', t) = 0 \quad \text{in } \mathbb{R}^N \times \mathbb{R}^+ \\ u(x, x', 0) = u_0(x, x') \quad \text{in } \mathbb{R}^N \end{cases},$$

is analyzed. In (8) $\alpha_1, \dots, \alpha_n$ are positive real coefficients, $N = n + m$, $(x, x') \in \mathbb{R}^n \times \mathbb{R}^m$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $x' = (x'_1, \dots, x'_m) \in \mathbb{R}^m$ and (D, D') denotes the gradient with respect to the variables in $\mathbb{R}^n \times \mathbb{R}^m$, with the position $D = (\partial_{x_1}, \dots, \partial_{x_n})$, $D' = (\partial_{x'_1}, \dots, \partial_{x'_m})$. A viscosity solution of (8) is given by the Hopf-Lax type formula:

$$(9) \quad u(x, x', t) =: Q_t^{(\alpha, 0)} u_0(x, x') \\ = \min_{(y, y') \in \mathbb{R}^N} \left\{ u_0(y, y') + \sum_{i=1}^n \frac{\alpha_i}{1 - e^{-2\alpha_i t}} (y_i - e^{-\alpha_i t} x_i)^2 + \frac{1}{2t} |x' - y'|^2 \right\}.$$

The hypercontractivity of the semigroup $(Q_t^{(\alpha, 0)})_{t \geq 0}$ and the optimality of such an estimate have been proved.

In this paper we consider the inequality established in [9]

$$(10) \quad q'(t) \mathcal{E}(e^{q(t) Q_t u_0(x)}) + N q(t) \alpha \int_{\mathbb{R}^N} e^{q(t) Q_t u_0(x)} dx \\ \leq \frac{q^2(t)}{2} \int_{\mathbb{R}^N} e^{q(t) Q_t u_0(x)} |D Q_t u_0(x)|^2 dx,$$

whose limit as t goes to 0 leads to the LSI (6) and we give some new relaxed conditions on the parameters in order to get (10). Moreover, here we show that a new LSI, direct consequence of the above reasoning

$$(11) \quad \mathcal{E}(u_0^2(x, x')) + n \int_{\mathbb{R}^N} u_0^2(x, x') dx dx' \\ \leq \frac{2}{\pi} \int_{\mathbb{R}^N} |Du_0(x, x')|^2 dx dx' + \frac{2}{\pi} \int_{\mathbb{R}^N} |D'u_0(x, x')|^2 dx dx',$$

is not optimal. Indeed that LSI (11) is weaker than (6), when we make $N = n + m$, $u_0(x, x')$ instead of $u_0(x)$, (D, D') instead of D . Here we discuss

the problem to generalize (11) and we are able to find the new inequality

$$\begin{aligned}
(12) \quad & \frac{q'(t)}{q(t)} \int_{\mathbb{R}^m} e^{q^*(t)Q_t^*v_0(x')} dx' \mathcal{E}_{dx}(e^{q(t)Q_t u_0(x)}) \\
& + \frac{q(t)(q^*)'(t)}{(q^*)^2(t)} \int_{\mathbb{R}^n} e^{q(t)Q_t u_0(x)} dx \mathcal{E}_{dx'}(e^{q^*(t)Q_t^*v_0(x')}) \\
& + \int_{\mathbb{R}^m} e^{q^*(t)Q_t^*v_0(x')} dx' \sum_{i=1}^n \alpha_i \int_{\mathbb{R}^n} e^{q(t)Q_t u_0(x)} dx \\
& \leq \frac{2}{q(t)} \int_{\mathbb{R}^m} e^{q^*(t)Q_t^*v_0(x')} dx' \int_{\mathbb{R}^n} \left| D e^{\frac{1}{2}q(t)Q_t u_0(x)} \right|^2 dx \\
& + \frac{2q(t)}{(q^*)^2(t)} \int_{\mathbb{R}^n} e^{q(t)Q_t u_0(x)} dx \int_{\mathbb{R}^m} \left| D' e^{\frac{1}{2}q^*(t)Q_t^*v_0(x')} \right|^2 dx'.
\end{aligned}$$

It is easy to see that, making $q(t) = q^*(t) = e^{\pi t}$, (12) reduces to (11) as t goes to 0. Nevertheless we are able to show the optimality of (12).

We prove it by means of the derivative's method of the norm function $F(t)$ of [5] and [6], used also for problem (5). Therefore, keeping in mind the structure of the semigroup $(Q_t^{(\alpha,0)})_{t \geq 0}$, we have decided to introduce a variant to the method of norm function $F(t)$ and to give a different role to the spaces \mathbb{R}^n and \mathbb{R}^m . There are several ways to make different roles for the spaces \mathbb{R}^n and \mathbb{R}^m : using different measures, or using different summability exponents, or both these variants. Here we follow the second approach.

On the other hand, we observe that optimality of the semigroup $Q_t^{(\alpha,0)}$ in [8] has been obtained by an initial datum $u_0(x) + v_0(x')$ where the variables split, and that

$$e^{Q_t^{(\alpha,0)}(u_0(x)+v_0(x'))} = e^{Q_t^\alpha(u_0(x))} e^{Q_t^*(v_0(x'))}.$$

For these reasons, here we use the function

$$\mathcal{G}(t) = \left\| e^{Q_t^\alpha(u_0(x))} \right\|_{L^{q(t)}(\mathbb{R}^n)} \left\| e^{Q_t^*(v_0(x'))} \right\|_{L^{q^*(t)}(\mathbb{R}^m)},$$

with $q(t)$ and $q^*(t)$ non decreasing C^1 functions. We observe that if $q(t) = q^*(t)$, the function $\mathcal{G}(t)$ coincides with $F(t)$. To see the generalization, we observe that inequality (12) can be written in this way

$$\begin{aligned}
& q'(t) \mathcal{E}_{dx}(e^{q(t)Q_t u_0(x)}) + q(t) \mathcal{I} \sum_{i=1}^n \alpha_i + \frac{q^2(t)}{(q^*)^2(t)} \frac{\mathcal{I}}{\mathcal{I}^*} (q^*)'(t) \mathcal{E}_{dx'}(e^{q^*(t)Q_t^*v_0(x')}) \\
& \leq 2 \int_{\mathbb{R}^n} \left| D e^{\frac{1}{2}q(t)Q_t u_0(x)} \right|^2 dx + \frac{2q^2(t)}{(q^*)^2(t)} \frac{\mathcal{I}}{\mathcal{I}^*} \int_{\mathbb{R}^m} \left| D' e^{\frac{1}{2}q^*(t)Q_t^*v_0(x')} \right|^2 dx',
\end{aligned}$$

with

$$\mathcal{I} = \int_{\mathbb{R}^n} e^{q(t)Q_t u_0(x)} dx, \quad \mathcal{I}^* = \int_{\mathbb{R}^m} e^{q^*(t)Q_t^* v_0(x')} dx'.$$

Hence, setting $q^*(t) = q(t) = e^{\pi t}$, (12) reduces to (11) as t goes to 0.

The paper is organized in four Sections. The Section 2 contains a revisited study of LSI and its optimality in the class of Hamilton-Jacobi equations we are interested in. Due to the particular case, some parameters can be estimated more efficiently than in [9]. Moreover, we establish a new LSI. In Section 3 the difficulty of studying the optimality of (12) is pointed out; we give an inequality which generalizes the previous (11) establishing a first result on the optimality problem, choosing appropriately the initial data. Finally, in Section 4 we describe some perspectives for the future.

2. Revised optimality.

2.1. Background.

Here we recall the notion of hypercontractivity, that will be used in our proof. It is a regularity property for semigroups.

Definition 2.1. A nonlinear semigroup $t \rightarrow Q_t$ is *hypercontractive* if there exists an increasing function $t \rightarrow q(t)$ such that for any function $u \in L^{q(0)}(\mathbb{R}^N)$:

$$\|e^{Q_t u}\|_{L^{q(t)}} \leq C(t) \|e^u\|_{L^{q(0)}}, \quad t \geq 0,$$

for some positive and continuous function $C(t) \leq 1$.

The following semigroup, introduced in [8],

$$(13) \quad Q_t^{(\alpha,0)} u_0(x, x') = \min_{(y, y') \in \mathbb{R}^N} \left\{ u_0(y, y') + \sum_{i=1}^n \frac{\alpha_i}{1 - e^{-2\alpha_i t}} (y_i - e^{-\alpha_i t} x_i)^2 + \frac{1}{2t} |x' - y'|^2 \right\},$$

is solution (in the viscosity sense) of the mixed problem (8) that we rewrite for convenience

$$\begin{cases} u_t(x, x', t) + \frac{1}{2} |Du(x, x', t)|^2 + \frac{1}{2} |D'u(x, x', t)|^2 + \\ + \sum_{i=1}^n \alpha_i x_i u_{x_i}(x, x', t) = 0 \quad \text{in } \mathbb{R}^N \times \mathbb{R}^+ \\ u(x, x', 0) = u_0(x, x') \quad \text{in } \mathbb{R}^N \end{cases},$$

$N = n + m$, $\alpha_1, \dots, \alpha_n$ positive coefficients, $(x, x') \in \mathbb{R}^n \times \mathbb{R}^m$, $D = (\partial_{x_1}, \dots, \partial_{x_n})$, $D' = (\partial_{x'_1}, \dots, \partial_{x'_m})$. The following result is here established. This is a simple consequence of the Theorem 3 in [9].

Proposition 2.1. *For any positive real number $p \leq 2$, and $\alpha < \pi < 2\alpha$ the semigroup defined by*

$$(14) \quad u(x, t) = \min_{y \in \mathbb{R}^N} \left[u_0(y) + \sum_{i=1}^N \frac{\alpha_i}{1 - e^{-2\alpha_i t}} (y_i - e^{-\alpha_i t} x_i)^2 \right],$$

is hypercontractive from $L^p(\mathbb{R}^N)$ to $L^{pe^{\alpha t}}(\mathbb{R}^N)$ and the following LSI holds true with $q(t) = pe^{\alpha t}$

$$(15) \quad q'(t)\mathcal{E}(e^{q(t)Q_t u_0(x)}) + Nq(t)\alpha \int_{\mathbb{R}^N} e^{q(t)Q_t u_0(x)} dx \\ \leq \frac{q^2(t)}{2} \int_{\mathbb{R}^N} e^{q(t)Q_t u_0(x)} |DQ_t u_0(x)|^2 dx.$$

Proof. Indeed following [8] the hypercontractivity holds

$$\|e^{Q_t u_0}\|_{L^{\omega e^{\alpha t}}(\mathbb{R}^N)} \leq \|e^{u_0}\|_{L^p(\mathbb{R}^N)}$$

for all the triple of real positive numbers (p, ω, t) for which $p \leq \omega$ and $t \in \mathbb{R}^+$ such that

$$(16) \quad \left(\frac{2\alpha \left(1 - \frac{p}{\omega} e^{-\alpha t}\right)}{1 - e^{-2\alpha t}} p \right)^{\frac{1}{2}} \leq \left(\int_{\mathbb{R}^N} e^{-\frac{1}{2}|x|^2} dx \right)^{\frac{1}{N}}$$

and in the proof of Theorem 3 in [9] it is used

$$\frac{1 - e^{-\beta t}}{1 - e^{-2\alpha_j t}} \leq 1,$$

Here, taking $\alpha_1 = \alpha_2 = \dots = \alpha_N =: \alpha$ and $\beta = \alpha$, since

$$\frac{1 - e^{-\alpha t}}{1 - e^{-2\alpha t}} = \frac{1}{1 + e^{-\alpha t}} \quad \text{and} \quad \int_{\mathbb{R}^N} e^{-\frac{1}{2}|x|^2} dx = (2\pi)^{\frac{N}{2}},$$

from (16) with $p = \omega$ we have

$$(17) \quad p \leq \frac{\pi}{\alpha} (1 + e^{-\alpha t}).$$

We select \tilde{t} such that

$$\frac{\pi}{\alpha} (1 + e^{-\alpha \tilde{t}}) = 2.$$

Solving in t we find

$$\tilde{t} = \tilde{t}(\alpha) = \frac{1}{\alpha} \log \frac{\pi}{\alpha - (\pi - \alpha)}.$$

By the assumption $\alpha < \pi < 2\alpha$ we see that

$$\frac{\pi}{\alpha - (\pi - \alpha)} > \frac{\pi}{\alpha} > 1.$$

Hence $\tilde{t} > 0$ and

$$\lim_{\alpha \rightarrow \pi^-} \tilde{t}(\alpha) = 0$$

In other words, we have shown that for any $\alpha < \pi < 2\alpha$, and for any $0 < t \leq \tilde{t}$, since $\frac{\pi}{\alpha}(1 + e^{-\alpha t}) \geq 2$, we are allowed to take $p = 2$ in (17). We observe that with respect to [9] the upper bound on p is relaxed.

Denoting by

$$F(t) = \|e^{Q_t u_0}\|_{L^{q(t)}(\mathbb{R}^N)},$$

with q a non decreasing C^1 function such that $q(0) = p$. We have that $F'(t) \leq 0$, $\forall t \in (0, \tilde{t})$, and the proof can be repeated with the same arguments in [8]. \square

2.2. Optimality.

Motivated by the above discussion (we observe that 2 is an admissible value for p), we select the function $q(t) = 2e^{\pi t}$. Letting $t \rightarrow 0^+$, we find

$$\pi \mathcal{E}(e^{2u_0(x)}) + N\alpha \int_{\mathbb{R}^N} e^{2u_0(x)} dx \leq \int_{\mathbb{R}^N} e^{2u_0(x)} |Du_0(x)|^2 dx.$$

Dividing by π and letting $\alpha \rightarrow \pi^-$, we have

$$\mathcal{E}(e^{2u_0(x)}) + N \int_{\mathbb{R}^N} e^{2u_0(x)} dx \leq \frac{1}{\pi} \int_{\mathbb{R}^N} e^{2u_0(x)} |Du_0(x)|^2 dx.$$

If we substitute in the above inequality the function

$$u_0(x) = -\frac{1}{2}A|x|^2, \quad A < \pi, \quad x \in \mathbb{R}^N,$$

we find:

$$\mathcal{E}(e^{2u_0(x)}) = \left(\frac{\pi}{A}\right)^{\frac{N}{2}} \left[-\frac{N}{2} - \frac{N}{2} \log \frac{\pi}{A}\right],$$

$$\int_{\mathbb{R}^N} e^{2u_0(x)} dx = \left(\frac{\pi}{A}\right)^{\frac{N}{2}},$$

$$\int_{\mathbb{R}^N} e^{2u_0(x)} |Du_0(x)|^2 dx = \int_{\mathbb{R}^N} |De^{u_0(x)}|^2 dx = \frac{AN}{2} \left(\frac{\pi}{A}\right)^{\frac{N}{2}}.$$

In conclusion, we have

$$\left(\frac{\pi}{A}\right)^{\frac{N}{2}} \left[-\frac{N}{2} - \frac{N}{2} \log \frac{\pi}{A}\right] + N \left(\frac{\pi}{A}\right)^{\frac{N}{2}} \leq \frac{AN}{2\pi} \left(\frac{\pi}{A}\right)^{\frac{N}{2}}.$$

Letting $A \rightarrow \pi^-$, we obtain the equality and therefore the optimality.

2.3. A new LSI.

Theorem 2.1. *Given an admissible initial datum u_0 , the following LSI holds true:*

$$(18) \quad \mathcal{E}(u_0^2(x, x')) + n \int_{\mathbb{R}^N} u_0^2(x, x') dx dx' \leq \frac{2}{\pi} \int_{\mathbb{R}^N} |Du_0(x, x')|^2 dx dx' + \frac{2}{\pi} \int_{\mathbb{R}^N} |D'u_0(x, x')|^2 dx dx'.$$

Proof. Given the semigroup $t \rightarrow Q_t^{(\alpha, 0)} u_0(x, x')$ previously defined in (9)

$$(19) \quad u(x, x', t) := Q_t^{(\alpha, 0)} u_0(x, x') = \min_{(y, y') \in \mathbb{R}^N} \left\{ u_0(y, y') + \sum_{i=1}^n \frac{\alpha_i}{1 - e^{-2\alpha_i t}} (y_i - e^{-\alpha_i t} x_i)^2 + \frac{1}{2t} |x' - y'|^2 \right\}$$

we introduce the norm function

$$F(t) = \left\| e^{Q_t^{(\alpha, 0)} u_0(x, x')} \right\|_{L^{q(t)}(\mathbb{R}^N)},$$

with q a non decreasing C^1 function, such that $q(0) = p$. The hypothesis that u_0 is an admissible function ensures

$$\|e^{u_0}\|_{L^p(\mathbb{R}^N)} < \infty.$$

We have

$$[F(t)]^{q(t)} = \int_{\mathbb{R}^N} e^{q(t)Q_t u_0} dx dx', \quad \log F(t) = \frac{1}{q(t)} \log \int_{\mathbb{R}^N} e^{q(t)Q_t u_0} dx dx'.$$

As usual, we differentiate in time F ; using a suitable choice of the function $q(t)$, $t \geq 0$, such that $q(0) = p$ and the hypercontractivity estimate, we obtain

$$(20) \quad q'(t) \mathcal{E}(e^{q(t)Q_t u_0}) + q(t) \sum_{i=1}^n \alpha_i \int_{\mathbb{R}^N} e^{q(t)Q_t u_0} dx dx' \leq 2 \int_{\mathbb{R}^N} \left| D e^{\frac{1}{2}q(t)Q_t u_0} \right|^2 dx dx' + 2 \int_{\mathbb{R}^N} \left| D' e^{\frac{1}{2}q(t)Q_t u_0} \right|^2 dx dx'.$$

Let us apply the above inequality setting:

$$h(x, x', t) = e^{\frac{1}{2}q(t)Q_t u_0} = e^{\frac{1}{2}e^{\pi t} Q_t u_0}$$

and

$$\alpha_i < \pi < 2\alpha_l, \quad i, l = 1, \dots, n,$$

in order to satisfy the hypothesis of Proposition 2.1. Thus:

$$\begin{aligned} \pi e^{\pi t} \mathcal{E}(h^2) + e^{\pi t} \left(\sum_{i=1}^n \alpha_i \right) \int_{\mathbb{R}^N} h^2 dx dx' \\ \leq 2 \int_{\mathbb{R}^N} |Dh|^2 dx dx' + 2 \int_{\mathbb{R}^N} |D'h|^2 dx dx'. \end{aligned}$$

Letting $\alpha_i \rightarrow \pi$ and $t \rightarrow 0^+$, we have

$$h = e^{\frac{1}{2} e^{\pi t} Q_t u_0} \rightarrow e^{\frac{u_0}{2}}.$$

Therefore:

$$\mathcal{E}(e^{u_0}) + n \int_{\mathbb{R}^N} e^{u_0} dx dx' \leq \frac{2}{\pi} \int_{\mathbb{R}^N} \left| D e^{\frac{u_0}{2}} \right|^2 dx dx' + \frac{2}{\pi} \int_{\mathbb{R}^N} \left| D' e^{\frac{u_0}{2}} \right|^2 dx dx',$$

then we obtain (11). \square

The analogous of inequality (10) in the case $N = n + m$ is

$$\begin{aligned} q'(t) \mathcal{E}(e^{q(t) Q_t u_0(x, x')}) + q(t) \sum_{i=1}^n \alpha_i \int_{\mathbb{R}^N} e^{q(t) Q_t u_0(x, x')} dx dx' \\ \leq \frac{q^2(t)}{2} \int_{\mathbb{R}^N} e^{q(t) Q_t u_0(x, x')} |D Q_t u_0(x, x')| dx dx' \\ + \frac{q^2(t)}{2} \int_{\mathbb{R}^N} e^{q(t) Q_t u_0(x, x')} |D' Q_t u_0(x, x')| dx dx' \\ = 2 \int_{\mathbb{R}^N} \left| D e^{q(t) Q_t u_0(x, x')} \right|^2 dx dx' + 2 \int_{\mathbb{R}^N} \left| D' e^{q(t) Q_t u_0(x, x')} \right|^2 dx dx'. \end{aligned}$$

In the same line of Section 2.2, we select the function $q(t) = 2e^{\pi t}$; we find the inequality

$$\begin{aligned} \mathcal{E}(e^{2u_0(x, x')}) + n \int_{\mathbb{R}^N} e^{2u_0(x, x')} dx dx' \\ \leq \frac{1}{\pi} \int_{\mathbb{R}^N} e^{2u_0(x, x')} |D u_0(x, x')|^2 dx dx' + \frac{1}{\pi} \int_{\mathbb{R}^N} e^{2u_0(x, x')} |D' u_0(x, x')|^2 dx dx'. \end{aligned}$$

Substituting in it the function $u_0(x, x') = -\frac{A}{2} |x|^2 - \frac{B}{2} |x'|^2$ and calculating the different terms, letting $A \rightarrow \pi^-$ and $B \rightarrow \pi^-$, we have

$$\frac{n}{2} - \frac{m}{2} \leq \frac{n}{2} + \frac{m}{2},$$

that is an equality if and only if $m = 0$.

3. An optimality result.

We need to introduce a variant to the method of the norm function $F(t)$. Actually, given the semigroups $Q_t u_0(x)$, $x \in \mathbb{R}^n$ and $Q_t^* v_0(x')$, $x' \in \mathbb{R}^m$, we introduce the function

$$\mathcal{G}(t) = \|e^{Q_t u_0}\|_{L^q(\mathbb{R}^n)} \cdot \|e^{Q_t^* v_0}\|_{L^{q^*}(\mathbb{R}^m)},$$

that is

$$\mathcal{G}(t) = \left(\int_{\mathbb{R}^n} e^{q(t)Q_t u_0(x)} dx \right)^{\frac{1}{q(t)}} \left(\int_{\mathbb{R}^m} e^{q^*(t)Q_t^* v_0(x')} dx' \right)^{\frac{1}{q^*(t)}},$$

with q and q^* non decreasing C^1 functions and u_0 and v_0 smooth enough. In this way, we give a different role to the spaces \mathbb{R}^n and \mathbb{R}^m by means of different summability exponents. Differentiating in time the function \mathcal{G} , we obtain

$$(21) \quad \frac{d}{dt} [\mathcal{G}(t)]^{q(t)} = e^{q(t) \log \mathcal{G}(t)} \left[q'(t) \log \mathcal{G}(t) + q(t) \frac{\mathcal{G}'(t)}{\mathcal{G}(t)} \right].$$

Setting

$$\begin{aligned} \mathcal{I} &= \int_{\mathbb{R}^n} e^{q(t)Q_t u_0(x)} dx, \\ \mathcal{I}^* &= \int_{\mathbb{R}^m} e^{q^*(t)Q_t^* v_0(x')} dx', \end{aligned}$$

we can write

$$\begin{aligned} \frac{d}{dt} [\mathcal{G}(t)]^{q(t)} &= \mathcal{I} \mathcal{I}^* \frac{q(t)}{q^*(t)} \left[\frac{q(t)}{q^*(t)} \right]' \log \mathcal{I}^* \\ &\quad + \mathcal{I}^* \frac{q(t)}{q^*(t)} \int_{\mathbb{R}^n} e^{q(t)Q_t u_0(x)} [q'(t)Q_t u_0(x) + q(t)\partial_t Q_t u_0(x)] dx \\ &+ \mathcal{I} \mathcal{I}^* \frac{q(t)}{q^*(t)} - 1 \frac{q(t)}{q^*(t)} \int_{\mathbb{R}^m} e^{q^*(t)Q_t^* v_0(x')} [(q^*)'(t)Q_t^* v_0(x') + q^*(t)\partial_t Q_t^* v_0(x')] dx'. \end{aligned}$$

Therefore

$$\begin{aligned} q(t)\mathcal{G}(t)^{q(t)-1}\mathcal{G}'(t) &= -\frac{q'(t)}{q(t)}\mathcal{G}(t)^{q(t)} \log \mathcal{G}(t)^{q(t)} + \mathcal{I} \mathcal{I}^* \frac{q(t)}{q^*(t)} \left[\frac{q(t)}{q^*(t)} \right]' \log \mathcal{I}^* \\ &\quad + \mathcal{I}^* \frac{q(t)}{q^*(t)} \int_{\mathbb{R}^n} e^{q(t)Q_t u_0(x)} [q'(t)Q_t u_0(x) + q(t)\partial_t Q_t u_0(x)] dx \\ &+ \mathcal{I} \mathcal{I}^* \frac{q(t)}{q^*(t)} - 1 \frac{q(t)}{q^*(t)} \int_{\mathbb{R}^m} e^{q^*(t)Q_t^* v_0(x')} [(q^*)'(t)Q_t^* v_0(x') + q^*(t)\partial_t Q_t^* v_0(x')] dx'. \end{aligned}$$

We observe that

$$\begin{aligned}\mathcal{G}(t)^{q(t)} &= \mathcal{I}\mathcal{I}^* \frac{q(t)}{q^*(t)}, \\ \log \mathcal{G}(t)^{q(t)} &= \log \mathcal{I} + \log \mathcal{I}^* \frac{q(t)}{q^*(t)}, \\ \mathcal{G}(t)^{q(t)} \log \mathcal{G}(t)^{q(t)} &= \mathcal{I}\mathcal{I}^* \frac{q(t)}{q^*(t)} \log \mathcal{I} + \mathcal{I}\mathcal{I}^* \frac{q(t)}{q^*(t)} \log \mathcal{I}^* \frac{q(t)}{q^*(t)}.\end{aligned}$$

Hence

$$\begin{aligned}q(t)\mathcal{G}(t)^{q(t)-1}\mathcal{G}'(t) &= -\frac{q'(t)}{q(t)}\mathcal{I}\mathcal{I}^* \frac{q(t)}{q^*(t)} \log \mathcal{I} - \frac{q'(t)}{q(t)}\mathcal{I}\mathcal{I}^* \frac{q(t)}{q^*(t)} \log \mathcal{I}^* \frac{q(t)}{q^*(t)} \\ &\quad + \mathcal{I}\mathcal{I}^* \frac{q(t)}{q^*(t)} \left[\frac{q(t)}{q^*(t)} \right]' \frac{q^*(t)}{q(t)} \log \mathcal{I}^* \frac{q(t)}{q^*(t)} \\ &\quad + \mathcal{I}^* \frac{q(t)}{q^*(t)} \frac{q'(t)}{q(t)} \int_{\mathbb{R}^n} e^{q(t)Q_t u_0(x)} q(t) Q_t u_0(x) dx \\ &\quad + \mathcal{I}^* \frac{q(t)}{q^*(t)} \int_{\mathbb{R}^n} e^{q(t)Q_t u_0(x)} q(t) \partial_t Q_t u_0(x) dx \\ &\quad + \mathcal{I}\mathcal{I}^* \frac{q(t)}{q^*(t)} - 1 \frac{q(t)(q^*)'(t)}{(q^*)^2(t)} \int_{\mathbb{R}^m} e^{q^*(t)Q_t^* v_0(x')} q^*(t) Q_t^* v_0(x') dx' \\ &\quad + \mathcal{I}\mathcal{I}^* \frac{q(t)}{q^*(t)} - 1 \int_{\mathbb{R}^m} e^{q^*(t)Q_t^* v_0(x')} q^*(t) \partial_t Q_t^* v_0(x') dx'.\end{aligned}$$

We notice that

$$\left[\frac{q(t)}{q^*(t)} \right]' \frac{q^*(t)}{q(t)} = \frac{q'(t)}{q(t)} - \frac{(q^*)'(t)}{q^*(t)},$$

that implies

$$\begin{aligned}q(t)\mathcal{G}(t)^{q(t)-1}\mathcal{G}'(t) &= \frac{q'(t)}{q(t)}\mathcal{I}^* \frac{q(t)}{q^*(t)} \mathcal{E}_{dx}(e^{q(t)Q_t u_0(x)}) \\ &\quad + \frac{q(t)(q^*)'(t)}{(q^*)^2(t)} \mathcal{I}\mathcal{I}^* \frac{q(t)}{q^*(t)} - 1 \mathcal{E}_{dx'}(e^{q^*(t)Q_t^* v_0(x')}) \\ &\quad + \mathcal{I}^* \frac{q(t)}{q^*(t)} \int_{\mathbb{R}^n} e^{q(t)Q_t u_0(x)} q(t) \partial_t Q_t u_0(x) dx \\ &\quad + \frac{q(t)}{q^*(t)} \mathcal{I}\mathcal{I}^* \frac{q(t)}{q^*(t)} - 1 \int_{\mathbb{R}^m} e^{q^*(t)Q_t^* v_0(x')} q^*(t) \partial_t Q_t^* v_0(x') dx'.\end{aligned}$$

Thus, since $\mathcal{G}'(t) \leq 0$ for t small enough

$$\begin{aligned} & \frac{q'(t)}{q(t)} \mathcal{I}^* \frac{q(t)}{q^*(t)} \mathcal{E}_{dx}(e^{q(t)Q_t u_0(x)}) + \frac{q(t)(q^*)'(t)}{(q^*)^2(t)} \mathcal{I} \mathcal{I}^* \frac{q(t)}{q^*(t)}^{-1} \mathcal{E}_{dx'}(e^{q^*(t)Q_t^* v_0(x')}) \\ & \quad + \mathcal{I}^* \frac{q(t)}{q^*(t)} \sum_{i=1}^n \alpha_i \int_{\mathbb{R}^n} e^{q(t)Q_t u_0(x)} dx \\ & \leq \frac{2}{q(t)} \mathcal{I}^* \frac{q(t)}{q^*(t)} \int_{\mathbb{R}^n} \left| D e^{\frac{1}{2}q(t)Q_t u_0(x)} \right|^2 dx \\ & \quad + \frac{2q(t)}{(q^*)^2(t)} \mathcal{I} \mathcal{I}^* \frac{q(t)}{q^*(t)}^{-1} \int_{\mathbb{R}^m} \left| D' e^{\frac{1}{2}q^*(t)Q_t^* v_0(x')} \right|^2 dx'. \end{aligned}$$

and finally, simplifying by $\mathcal{I}^* \frac{q(t)}{q^*(t)}^{-1}$, we have

$$\begin{aligned} & \frac{q'(t)}{q(t)} \int_{\mathbb{R}^m} e^{q^*(t)Q_t^* v_0(x')} dx' \mathcal{E}_{dx}(e^{q(t)Q_t u_0(x)}) \\ & \quad + \frac{q(t)(q^*)'(t)}{(q^*)^2(t)} \int_{\mathbb{R}^n} e^{q(t)Q_t u_0(x)} dx \mathcal{E}_{dx'}(e^{q^*(t)Q_t^* v_0(x')}) \\ & \quad + \int_{\mathbb{R}^m} e^{q^*(t)Q_t^* v_0(x')} dx' \sum_{i=1}^n \alpha_i \int_{\mathbb{R}^n} e^{q(t)Q_t u_0(x)} dx \\ & \leq \frac{2}{q(t)} \int_{\mathbb{R}^m} e^{q^*(t)Q_t^* v_0(x')} dx' \int_{\mathbb{R}^n} \left| D e^{\frac{1}{2}q(t)Q_t u_0(x)} \right|^2 dx \\ & \quad + \frac{2q(t)}{(q^*)^2(t)} \int_{\mathbb{R}^n} e^{q(t)Q_t u_0(x)} dx \int_{\mathbb{R}^m} \left| D' e^{\frac{1}{2}q^*(t)Q_t^* v_0(x')} \right|^2 dx'. \end{aligned}$$

We can write the above inequality as

$$(22) \quad \begin{aligned} & \frac{1}{q(t)} \mathcal{I}^* \left[q'(t) \mathcal{E}_{dx}(e^{q(t)Q_t u_0(x)}) + q(t) \mathcal{I} \sum_{i=1}^n \alpha_i - 2 \int_{\mathbb{R}^n} \left| D e^{\frac{1}{2}q(t)Q_t u_0(x)} \right|^2 dx \right] \\ & + \frac{q(t)}{(q^*)^2(t)} \mathcal{I} \left[(q^*)'(t) \mathcal{E}_{dx'}(e^{q^*(t)Q_t^* v_0(x')}) - 2 \int_{\mathbb{R}^m} \left| D' e^{\frac{1}{2}q^*(t)Q_t^* v_0(x')} \right|^2 dx' \right] \leq 0, \end{aligned}$$

$$\begin{aligned} & q'(t) \mathcal{E}_{dx}(e^{q(t)Q_t u_0(x)}) + q(t) \mathcal{I} \sum_{i=1}^n \alpha_i + \frac{q^2(t)}{(q^*)^2(t)} \frac{\mathcal{I}}{\mathcal{I}^*} (q^*)'(t) \mathcal{E}_{dx'}(e^{q^*(t)Q_t^* v_0(x')}) \\ & \leq 2 \int_{\mathbb{R}^n} \left| D e^{\frac{1}{2}q(t)Q_t u_0(x)} \right|^2 dx + \frac{2q^2(t)}{(q^*)^2(t)} \frac{\mathcal{I}}{\mathcal{I}^*} \int_{\mathbb{R}^m} \left| D' e^{\frac{1}{2}q^*(t)Q_t^* v_0(x')} \right|^2 dx'. \end{aligned}$$

From such inequality it is easy to state:

- a) if $u_0(x)$ is an initial datum which optimizes the LSI (10) for $n = N$, then (22) becomes the LSI (1) for $N = m$;
- b) if $v_0(x')$ is an initial datum which optimizes the LSI (1) for $N = m$, then (22) becomes the LSI (10) with $N = n$;
- c) if $u_0(x)$ and $v_0(x')$ are initial data which optimize (10) and (1) respectively, the initial datum $u_0(x, x') = u_0(x) + v_0(x')$ optimize the inequality (22).

But now we want to prove that there are several initial data optimizing (22), which are different from those described in c), for t, p, p^* beyond u_0 and v_0 . This is shown in the following section.

3.1. A particular case.

We discuss the optimality with the setting

$$u_0(x) = -A|x|^2, \quad v_0(x') = -B|x'|^2, \quad \alpha_1 = \dots = \alpha_n = \alpha > 0.$$

In our particular case, with $B = \frac{\alpha}{2}$, the solution of problem (8) is

$$u(x, x', t) = -\tilde{A}|x|^2 - \tilde{B}|x'|^2,$$

where

$$\tilde{A}(t) = \frac{A\alpha e^{-2\alpha t}}{\alpha - A(1 - e^{-2\alpha t})}, \quad \tilde{B}(t) = \frac{\alpha}{2(1 - \alpha t)},$$

with $0 < A < \alpha$. In order to show the equality in (12), we compute the quantities appearing there

$$\int_{\mathbb{R}^n} e^{q(t)Q_t u_0(x)} dx = \left[\frac{\pi}{q(t)\tilde{A}(t)} \right]^{\frac{n}{2}}, \quad \int_{\mathbb{R}^m} e^{q^*(t)Q_t^* v_0(x')} dx' = \left[\frac{\pi}{q^*(t)\tilde{B}(t)} \right]^{\frac{m}{2}}.$$

By (7), we have

$$\mathcal{E}_{dx}(e^{q(t)Q_t u_0(x)}) = -\frac{n}{2} \left[\frac{\pi}{q(t)\tilde{A}(t)} \right]^{\frac{n}{2}} - \frac{n}{2} \log \left[\frac{\pi}{q(t)\tilde{A}(t)} \right] \left[\frac{\pi}{q(t)\tilde{A}(t)} \right]^{\frac{n}{2}},$$

$$\mathcal{E}_{dx'}(e^{q^*(t)Q_t^* v_0(x')}) = -\frac{m}{2} \left[\frac{\pi}{q^*(t)\tilde{B}(t)} \right]^{\frac{m}{2}} - \frac{m}{2} \log \left[\frac{\pi}{q^*(t)\tilde{B}(t)} \right] \left[\frac{\pi}{q^*(t)\tilde{B}(t)} \right]^{\frac{m}{2}}.$$

Next, we calculate the the integrals in the right hand side of (12)

$$\int_{\mathbb{R}^n} \left| D e^{\frac{1}{2}q(t)Q_t u_0(x)} \right|^2 dx = \frac{n}{2} q(t)\tilde{A}(t) \left[\frac{\pi}{q(t)\tilde{A}(t)} \right]^{\frac{n}{2}},$$

$$\int_{\mathbb{R}^m} \left| D' e^{\frac{1}{2} q^*(t) Q_t^* v_0(x')} \right|^2 dx' = \frac{m}{2} q^*(t) \tilde{B}(t) \left[\frac{\pi}{q^*(t) \tilde{B}(t)} \right]^{\frac{m}{2}}.$$

Finally, we need to choose properly the functions q and q^* . We fix

$$q(t) = p e^{\alpha t}, \quad q^*(t) = p^* e^{\beta t}.$$

Substituting in (12), taking into account the computations and

$$\frac{q'(t)}{q(t)} = \alpha, \quad \frac{(q^*)'(t)}{q^*(t)} = \beta,$$

we have

$$(23) \quad n\alpha \leq n \left[\frac{1}{2}\alpha + \frac{1}{2}\alpha \log \left(\frac{\pi}{p e^{\alpha t} \tilde{A}(t)} \right) + \tilde{A}(t) - \alpha \right] + n\alpha \\ + m \frac{p}{p^*} e^{(\alpha-\beta)t} \left[\tilde{B}(t) + \frac{1}{2}\beta \left(1 + \log \left(\frac{\pi}{p^* e^{\beta t} \tilde{B}(t)} \right) \right) \right].$$

Proposition 3.1. *Given the initial datum*

$$u_0(x) + v_0(x') = -A|x|^2 - \frac{\alpha}{2}|x'|^2,$$

it is possible to find t^ , p , p^* and β such that (23) becomes an equality.*

Proof. In order to get an equality in (23), we impose the following conditions

$$\begin{cases} 1 + \log \left(\frac{\pi}{p e^{\alpha t} \tilde{A}(t)} + \frac{2}{\alpha} \tilde{A}(t) - 2 \right) = 0; \\ \tilde{B}(t) + \frac{1}{2}\beta \log \left(\frac{\pi e}{p^* e^{\beta t} \tilde{B}(t)} \right) = 0. \end{cases}$$

We discuss

$$(24) \quad 1 + \log \left(\frac{\pi}{p e^{\alpha t} \tilde{A}(t)} + \frac{2}{\alpha} \tilde{A}(t) - 2 \right) = 0.$$

Setting

$$(25) \quad p = \frac{\pi}{\tilde{A}(t)},$$

we obtain

$$(26) \quad \tilde{A}(t) - \frac{\alpha^2}{2}t - \frac{\alpha}{2} = 0.$$

This means that

$$\tilde{A}(t) = \frac{A\alpha e^{-2\alpha t}}{\alpha - A(1 - e^{-2\alpha t})} = \frac{\alpha^2}{2}t + \frac{\alpha}{2}.$$

In Figure 1, we plot the graphics of the two functions for the particular choice $\alpha = 2$ and $A = \frac{3}{2}$. Observe that $\tilde{A}(t)$ is decreasing and

$$\tilde{A}(0) = A < \alpha, \quad \lim_{t \rightarrow \infty} \tilde{A}(t) = 0.$$

Say t^* the unique positive solution of (26). Notice that, since

$$\tilde{A} < \alpha,$$

by (26) $t^* < \frac{1}{\alpha}$ and by (25) $p > \frac{\pi}{\alpha}$. Moreover

$$Ae^{-2\alpha t} < \tilde{A}(t),$$

and

$$p < \frac{\pi}{A}e^{2\alpha t^*}.$$

This means that as A is close to α and α is close to π , the value of p is close to 1.

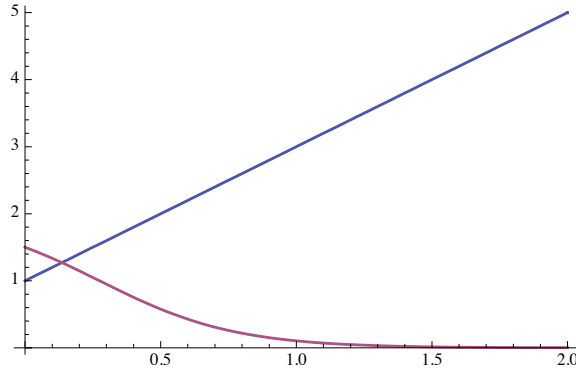


Figure 1. Plot of the functions $f(t) = 1 + 2t$ and $f(t) = \frac{3}{\frac{1}{2}e^{4t} + \frac{3}{2}}$.

The third equation gives

$$p^* = \frac{\pi}{\tilde{B}(t)} e^{1 - \beta t^* + \frac{2\tilde{B}(t^*)}{\beta}}.$$

This leads to select a positive real number β such that

$$1 - \beta t + \frac{2\tilde{B}(t)}{\beta} = 0,$$

therefore

$$\beta = \frac{1 + \sqrt{1 + 8\tilde{B}(t^*)}}{2t^*}.$$

Observe that $\beta > \alpha$. Hence we have proved the theorem. \square

4. Perspectives for the future.

A further study could be to obtain a LSI, passing to the limit on the time variable, and to define a more general entropy function dealing with functions depending on both the variables x, x' . Both the reserach lines require further investigations and are in progress.

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