Communications in Applied and Industrial Mathematics, ISSN 2038-0909, e-425
DOI: 10.1685/journal.caim.425

A Ginzburg-Landau model in superfluidity

Alessia Berti[1], Valeria Berti[2]

[1] *Facoltà di Ingegneria,*
Università e-Campus, Italy
alessia.berti@ing.unibs.it

[2] *Dipartimento di Matematica,*
Università degli Studi di Bologna, Italy
valeria.berti@unibo.it

Communicated by Nicola Bellomo

Abstract

A dynamical model for the normal-superfluid transition in ^{4}He, accounting for variations of temperature and pressure, is proposed. The phenomenon is analyzed in the framework of Ginzburg-Landau theory characterizing the phase of the material with an order parameter related to the concentration of the superfluid particles. The constitutive choices allow one to prove the existence of a critical velocity above which superfluid properties disappear and to recover the usual phase diagram of liquid helium. The thermodynamic restrictions in the constitutive relations which ensure the Clausius-Duhem inequality have been pointed out.

Keywords: superfluids, second-order phase transitions, Ginzburg-Landau equation, thermodynamics.

AMS subject classification: 82D50, 74A15, 82C26.

1. Introduction.

Immediately below its boiling temperature $\theta = 4.21K$, an isotope of helium, ^{4}He, behaves like an ordinary fluid with small viscosity. This state is called He I-phase. However, when the temperature is lowered under the critical value $\theta_\lambda = 2.17K$, called $\lambda-$point, ^{4}He undergoes a phase transition and it is able to flow without viscosity in narrow capillaries. This phase is called He II. The passage from He I to He II is a second order phase transition, since no latent heat is released or absorbed. The phase diagram of the substance is represented in Figure 1.

The traditional theories of superfluidity are based on the two-fluid model, proposed first by Tisza [1], that considers a superfluid as a mixture of two fluids: normal fluid and superfluid. According to this theory,

Received on October 28th, 2012. Accepted on February 21st, 2013. Published on February 28th, 2013.

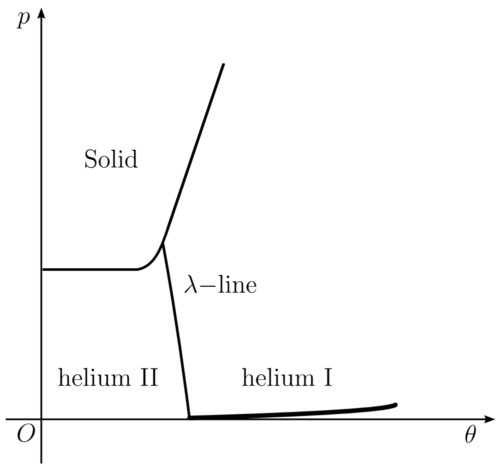

Figure 1. Phase diagram of liquid helium.

the density of the fluid is the sum of a normal and a superfluid component

$$\rho = \rho_n + \rho_s$$

and the total current density is given by

$$\mathbf{j} = \rho\mathbf{v} = \rho_n\mathbf{v}_n + \rho_s\mathbf{v}_s,$$

where $\mathbf{v}_n$ and $\mathbf{v}_s$ are called respectively the normal and the superfluid velocity. A similar point of view is adopted by Landau [2], by assuming that each particle of the superfluid behaves as a pair endowed with two kinds of excitations at the same instant, represented by the velocities $\mathbf{v}_n$ and $\mathbf{v}_s$.

More recently in [3] the author analyzes the normal-superfluid phase transition occurring in ^{4}He in the framework of the Ginzburg-Landau theory, by means of a model which emphasizes the analogies between superfluidity and superconductivity [4,5]. The superfluid component $\mathbf{v}_s$ is supposed to satisfy an evolution equation similar to the differential equation governing the motion of the superconducting electrons inside a superconductor [6].

In our paper we consider a generalization of the latter model, by keeping into account variations of the mass density of the fluid. Moreover, we assume that the normal component is a compressible fluid. Accordingly, the pressure becomes a new variable of the problem and we are able to recover the λ−line of the phase diagram in Figure 1.

Our model includes an evolutive equation for the temperature obtained from an appropriate form of the energy balance and it is compatible with the second law of Thermodynamics.

DOI: 10.1685/journal.caim.425

Finally, we introduce a new set of variables which allow a more direct comparison with the classical Ginzburg-Landau model of superconductivity.

2. Evolution equations for phase and velocity.

The Ginzburg-Landau theory of phase transitions is based on the introduction of a phase variable, also called order parameter, providing a measure of the internal order structure of the material. Since the superfluid phase is considered a more "ordered" state than the normal one [2], the order parameter is related to the concentration of the superfluid particles. More precisely, we define a scalar variable $\varphi \in [-1, 1]$, such that φ^2 is the density of the superfluid particles and $\varphi^2 = 0$ in the normal phase, while $\varphi^2 > 0$ in the superfluid phase.

As it is customary in other phase transition models [6,7], the differential equation governing the evolution of φ can be interpreted as a balance law on the internal order structure, namely

$$\rho k = \nabla \cdot \mathbf{p},$$

where

$$k = \tau\varphi_t + \theta_\lambda F'(\varphi) + mG'(\varphi)$$
$$\mathbf{p} = \frac{\rho}{\kappa^2}\nabla\varphi.$$

Here the subscript t stands for the time derivative, $m > 0$ is a suitable coefficient depending on the variables that induce the transition, the potentials F and G characterize the order of the transition and κ, τ are positive constants. Accordingly, the evolution equation for the order parameter reads

$$\tau\rho\varphi_t = \frac{1}{\kappa^2}\nabla \cdot (\rho\nabla\varphi) - \rho\theta_\lambda F'(\varphi) - \rho m G'(\varphi).$$

The typical choice characterizing second-order phase transitions, which is adopted in the classical Ginzburg-Landau theory of superconductivity [8], is

$$F(\varphi) = \frac{\varphi^4}{4} - \frac{\varphi^2}{2}, \qquad G(\varphi) = \frac{\varphi^2}{2}.$$

Therefore the function

$$W(\varphi) = \theta_\lambda F(\varphi) + mG(\varphi)$$

admits its minimum value at $\varphi = 0$ when $m \geq \theta_\lambda$ and at $\varphi = \pm\tilde{\varphi}(m)$, $0 < \tilde{\varphi}(m) < 1$, when $0 < m < \theta_\lambda$. This proves that the transition occurs when $m = \theta_\lambda$ and that $m > \theta_\lambda$ in the normal phase.

The main assumption of our model is the decomposition of the velocity of the superfluid into two components as

$$\mathbf{v} = \mathbf{v}_n + \varphi^2 \mathbf{v}_s. \tag{1}$$

This is consistent with the Landau's point of view. Indeed, when $\varphi = 0$, the fluid is in the normal state and the velocity coincides with the normal component $\mathbf{v}_n$. Conversely, if $\varphi^2 > 0$ both the two components (normal and superfluid) are present, so that the particles of the fluid are endowed with two simultaneous excitations.

The choice (1) is also suggested by the analogy with the model of superconductivity, in which the total electric current density is given by

$$\mathbf{J} = \mathbf{J}_n + \mathbf{J}_s,$$

where $\mathbf{J}_n$ is the normal current and $\mathbf{J}_s$ is the current due to the superconducting electrons. The latter is related to the velocity of the superconducting electrons by the expression $\mathbf{J}_s = \varphi^2 \mathbf{v}_s$ [6].

Following [3], in our model we assume

$$m = \theta + \lambda p + \mathbf{v}_s^2, \tag{2}$$

where p is the pressure and $\lambda \geq 0$ is a constant. Such a choice allows us to prove the existence of a critical velocity, depending on temperature and pressure, above which superfluid properties disappear. Indeed if $m = \theta + \lambda p + \mathbf{v}_s^2 > \theta_\lambda$, the fluid is in the normal phase.

Moreover, in steady conditions when $\mathbf{v}_s = 0$, relation (2) implies that the transition occurs if

$$\theta + \lambda p = \theta_\lambda,$$

which is a good approximation of the $\lambda-$line shown in the phase diagram of Figure 1. In the previous model for superfluidity proposed in [3], the coefficient λ was supposed to vanish, which corresponds to approximate the $\lambda-$line with a vertical line. Here, we consider a more general model where $\lambda \geq 0$.

Accordingly, the evolution equation of the phase involves the pressure and reads

$$\tau \rho \varphi_t = \frac{1}{\kappa^2} \nabla \cdot (\rho \nabla \varphi) - \rho \theta_\lambda (\varphi^2 - 1)\varphi - \rho(\theta + \lambda p + \mathbf{v}_s^2)\varphi. \tag{3}$$

We complete the equation with the boundary and initial conditions, typical of the Ginzburg-Landau equation

$$\nabla \varphi \cdot \mathbf{n}|_{\partial\Omega} = 0, \qquad \varphi(x,0) = \varphi_0(x).$$

DOI: 10.1685/journal.caim.425

The differential equations ruling the motion of the superfluid component are chosen in a similar way of the ones governing the evolution of the superconducting electrons inside a superconductor, *i.e.*

$$\partial_t \mathbf{v}_s = -\nabla\phi_s - \nabla\times\mathbf{v}_n - \rho\varphi^2\mathbf{v}_s + \nabla\theta \tag{4}$$

$$\nabla\cdot(\rho\varphi^2\mathbf{v}_s) = -\tau\kappa^2\rho\varphi^2\phi_s, \tag{5}$$

where ϕ_s is a suitable scalar function referable to a "pressure" due to the superfluid component. We associate to these equations the boundary conditions

$$\mathbf{v}_s\cdot\mathbf{n}|_{\partial\Omega} = 0, \tag{6}$$

$$(\nabla\times\mathbf{v}_s)\times\mathbf{n}|_{\partial\Omega} = \omega, \tag{7}$$

where ω is a known function.

For the velocity $\mathbf{v}_n$ we propose the following equation

$$\rho\partial_t\mathbf{v}_n = \nabla\times\partial_t\mathbf{v}_s + \nu\nabla(\nabla\cdot\mathbf{v}_n) - \nabla p + \rho\mathbf{g}, \tag{8}$$

where $\nu > 0$ is the viscosity coefficient and $\mathbf{g}$ denotes the external force density.

By applying the curl operator to equation (4), we obtain

$$\nabla\times\partial_t\mathbf{v}_s = -\nabla\times\nabla\times\mathbf{v}_n - \nabla\times(\rho\varphi^2\mathbf{v}_s).$$

A substitution into equation (8) leads to differential equation

$$\rho\partial_t\mathbf{v}_n + \nabla\times\nabla\times\mathbf{v}_n - \nu\nabla(\nabla\cdot\mathbf{v}_n) = -\nabla p - \nabla\times(\rho\varphi^2\mathbf{v}_s) + \rho\mathbf{g}.$$

Notice that, when the fluid is in the normal state, the previous equation reduces to the linearized Navier-Stokes equation.

In this paper, we assume that the normal component behaves like a compressible fluid, satisfying the relation

$$\nabla\cdot\mathbf{v}_n = \lambda\rho\varphi\varphi_t. \tag{9}$$

The latter generalizes the model in [3], ensuring the incompressibility of the normal fluid if $\lambda = 0$.

We append to (8) the usual boundary condition:

$$\mathbf{v}_n|_{\partial\Omega} = \mathbf{0}.$$

The velocity $\mathbf{v}$ is related to the density ρ by means of the continuity equation

$$\rho_t + \nabla\cdot(\rho\mathbf{v}) = 0.$$

In view of the representation (1), we obtain

$$\rho_t + \nabla\cdot(\rho\mathbf{v}_n + \rho\varphi^2\mathbf{v}_s) = 0.$$

3. Heat equation and thermodynamical consistence.

The evolution equation for the temperature is deduced from energy balance law [7]

$$\rho E_t = \mathcal{P}^i_m + \mathcal{P}^i_\varphi + \rho h, \tag{10}$$

where E is the total energy, $\mathcal{P}^i_m$ is the internal mechanical power, $\mathcal{P}^i_\varphi$ is the internal power due to the order parameter and h is the thermal power.

Hereafter, we take the assumption of slow motions, *i.e.* we assume that the material time derivative coincides with the partial time derivative. The general case will be investigated in our future work.

The explicit expression of the internal power $\mathcal{P}^i_\varphi$ is obtained by multiplying equation (3) by φ_t, that is

$$\tau\rho\varphi_t^2 - \frac{1}{\kappa^2}\nabla\cdot(\rho\nabla\varphi)\varphi_t + \rho\theta_\lambda(\varphi^3-\varphi)\varphi_t + \rho(\theta+\lambda p+\mathbf{v}_s^2)\varphi\varphi_t = 0.$$

By using the relation (9), the previous equation expresses the balance between the internal and external powers related to the phase variable

$$\mathcal{P}^i_\varphi = \mathcal{P}^e_\varphi,$$

where the internal and external powers are defined by

$$\begin{aligned}\mathcal{P}^i_\varphi &= \tau\rho\varphi_t^2 + \rho\left[\frac{1}{2\kappa^2}|\nabla\varphi|^2 + \theta_\lambda\left(\frac{1}{4}\varphi^4 - \frac{1}{2}\varphi^2\right)\right]_t + \rho(\theta+\mathbf{v}_s^2)\varphi\varphi_t\\ &\quad -\mathbf{v}_n\cdot\nabla p\\ \mathcal{P}^e_\varphi &= \nabla\cdot\left(\frac{1}{\kappa^2}\rho\varphi_t\nabla\varphi - p\mathbf{v}_n\right).\end{aligned}$$

Similarly the balance of the mechanical powers is obtained by multiplying equation (4) by $\partial_t\mathbf{v}_s + \nabla\phi_s - \nabla\theta$, (8) by $\mathbf{v}_n$ and adding the resulting equations. We deduce the relation

$$\begin{aligned}&|\partial_t\mathbf{v}_s + \nabla\phi_s - \nabla\theta|^2 + \rho\varphi^2\mathbf{v}_s\cdot(\partial_t\mathbf{v}_s + \nabla\phi_s - \nabla\theta) + \nabla\times\mathbf{v}_n\cdot(\nabla\phi_s - \nabla\theta)\\ &+\frac{\rho}{2}\partial_t\mathbf{v}_n^2 + [-\nu\nabla(\nabla\cdot\mathbf{v}_n) + \nabla p - \rho\mathbf{g}]\cdot\mathbf{v}_n = 0.\end{aligned}$$

By means of (5), we obtain

$$\begin{aligned}&|\partial_t\mathbf{v}_s + \nabla\phi_s - \nabla\theta|^2 + \rho\varphi^2\mathbf{v}_s\cdot(\partial_t\mathbf{v}_s - \nabla\theta) + \tau\kappa^2\rho\varphi^2\phi_s^2 + \frac{\rho}{2}\partial_t\mathbf{v}_n^2\\ &+\nu|\nabla\cdot\mathbf{v}_n|^2 + \nabla p\cdot\mathbf{v}_n =\\ &-\nabla\cdot[\mathbf{v}_n\times\nabla\phi_s - \mathbf{v}_n\times\nabla\theta + \rho\varphi^2\phi_s\mathbf{v}_s - (\nabla\cdot\mathbf{v}_n)\mathbf{v}_n] + \rho\mathbf{g}\cdot\mathbf{v}_n.\end{aligned}$$

We identify the left hand-side and the right hand side of the previous equation with the internal and external mechanical power respectively.

Hence the total internal power is given by

$$\begin{aligned}\mathcal{P}^i &= \mathcal{P}^i_\varphi + \mathcal{P}^i_m \\ &= \rho\left[\frac{1}{2\kappa^2}|\nabla\varphi|^2 + \theta_\lambda\left(\frac{1}{4}\varphi^4 - \frac{1}{2}\varphi^2\right) + \frac{1}{2}\varphi^2\mathbf{v}_s^2 + \frac{1}{2}\mathbf{v}_n^2\right]_t + \tau\rho\varphi_t^2 + \rho\theta\varphi\varphi_t \\ &\quad + |\partial_t\mathbf{v}_s + \nabla\phi_s - \nabla\theta|^2 - \rho\varphi^2\mathbf{v}_s\cdot\nabla\theta + \tau\kappa^2\rho\varphi^2\phi_s^2 + \nu|\nabla\cdot\mathbf{v}_n|^2. \qquad (11)\end{aligned}$$

We assume that the total energy E is the function

$$E = \frac{1}{2\kappa^2}|\nabla\varphi|^2 + \theta_\lambda F(\varphi) + e_0(\theta) + \frac{1}{2}\varphi^2\mathbf{v}_s^2 + \frac{1}{2}\mathbf{v}_n^2,$$

where $e_0(\theta)$ depends only on the temperature. More precisely, e_0' represents the specific heat of ^{4}He whose plot is depicted in Figure 2 according to experimental data.

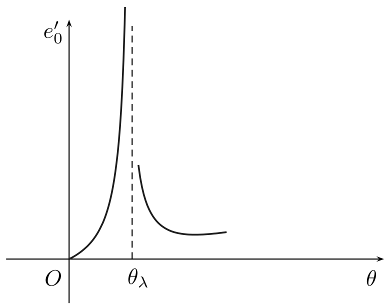

Figure 2. Specific heat of liquid helium.

Substitution into (10) provides

$$\begin{aligned}\rho h = \rho e_0'(\theta)\theta_t - \tau\rho\varphi_t^2 - \rho\theta\varphi\varphi_t - |\partial_t\mathbf{v}_s + \nabla\phi_s - \nabla\theta|^2 \\ +\rho\varphi^2\mathbf{v}_s\cdot\nabla\theta - \tau\kappa^2\rho\varphi^2\phi_s^2 - \nu|\nabla\cdot\mathbf{v}_n|^2. \qquad (12)\end{aligned}$$

The evolution equation of the variable θ is obtained by substitution of the thermal power in the heat equation

$$\rho h = -\nabla\cdot\mathbf{q} + \rho r, \qquad (13)$$

where $\mathbf{q}$ is the heat flux and r is the heat supply. For the heat flux we assume the following constitutive equation

$$\mathbf{q} = -k_0(\theta)\nabla\theta - \rho\varphi^2\theta\mathbf{v}_s, \qquad (14)$$

where $k_0(\theta)$ denotes the thermal conductivity. Notice that, when the fluid is in the normal phase, *i.e.* $\varphi = 0$, equation (14) reduces to the usual Fourier law. On the contrary, in the superfluid state, the superfluid component of the velocity $\mathbf{v}_s$ is related to the heat flux inside the material.

Equation (14) is not dissimilar to the equation proposed in the two-fluid model (see for instance [1,2]). Our approach differs from the model based on the extended thermodynamics where the heat flux is an independent variable satisfying a Cattaneo-Maxwell equation [9]. For a comparison between the two points of view, see [10,11].

Owing to (12), from (13) and (14) we obtain

$$\begin{aligned}\rho e_0'(\theta)\theta_t &= \tau\rho\varphi_t^2 + \rho\theta\varphi\varphi_t + |\partial_t \mathbf{v}_s + \nabla\phi_s - \nabla\theta|^2 + \nu|\nabla\cdot\mathbf{v}_n|^2\\ &+\tau\kappa^2\rho\varphi^2\phi_s^2 + \nabla\cdot(k_0(\theta)\nabla\theta) + \nabla\cdot(\rho\varphi^2\mathbf{v}_s)\theta + \rho r.\end{aligned}$$

We append to the previous equation the boundary condition

$$\mathbf{q}\cdot\mathbf{n}|_{\partial\Omega} = 0,$$

which, in view of (6) and (14), yields

$$\nabla\theta\cdot\mathbf{n}|_{\partial\Omega} = 0.$$

Now we prove that our model is consistent with the second law of Thermodynamics written by means of the Clausius-Duhem inequality as

$$\rho\eta_t \geq -\nabla\cdot\left(\frac{\mathbf{q}}{\theta}\right) + \frac{\rho r}{\theta},$$

where η is the entropy.

By means of (10) we deduce that

$$\rho\eta_t\theta \geq \rho h + \frac{1}{\theta}\mathbf{q}\cdot\nabla\theta = \rho E_t - \mathcal{P}_\varphi^i - \mathcal{P}_m^i + \frac{1}{\theta}\mathbf{q}\cdot\nabla\theta.$$

Substitution of (11) and (14) yields the inequality

$$\begin{aligned}\rho\eta_t\theta &\geq \rho e_0'(\theta)\theta_t - \rho\theta\varphi\varphi_t - \tau\rho\varphi_t^2 - |\partial_t\mathbf{v}_s + \nabla\phi_s - \nabla\theta|^2\\ &-\tau\kappa^2\rho\varphi^2\phi_s^2 - \nu|\nabla\cdot\mathbf{v}_n|^2 - \frac{k_0(\theta)}{\theta}|\nabla\theta|^2.\end{aligned} \tag{15}$$

This relation suggests the following choice of the entropy

$$\eta = \eta(\varphi,\theta) = -\frac{1}{2}\varphi^2 + \alpha(\theta), \tag{16}$$

where $\alpha(\theta)$ is defined as

$$\alpha(\theta) = \int \frac{e_0'(\theta)}{\theta} d\theta.$$

Hence (15) reduces to

$$\tau\rho\varphi_t^2 + |\partial_t \mathbf{v}_s + \nabla\phi_s - \nabla\theta|^2 + \tau\kappa^2\rho\varphi^2\phi_s^2 + \nu|\nabla\cdot\mathbf{v}_n|^2 + \frac{k_0(\theta)}{\theta}|\nabla\theta|^2 \geq 0.$$

Therefore, we obtain the following restrictions which guarantee the thermodynamical compatibility of our model.

Theorem 3.1. *Let η be defined as in (16). Then the Clausius-Duhem inequality is satisfied if*

$$\tau \geq 0, \quad \nu \geq 0, \quad k_0(\theta) \geq 0,$$

for any θ.

The expression of the entropy allow us to verify that the passage from the normal to superfluid phase is a second-order transition. Indeed, the latent heat ℓ is given by

$$\ell = \theta_\lambda[\eta(\tilde{\varphi}(\theta_\lambda), \theta_\lambda) - \eta(0, \theta_\lambda)]$$

where $\tilde{\varphi}(\theta)$ is the local minimum of the function $W(\varphi)$. In this case, since $\tilde{\varphi}(\theta_\lambda) = 0$ the latent heat vanishes.

4. Gauge invariance.

The model we have developed in the previous sections is ruled by the following system of differential equations

$$\tau\rho\varphi_t = \frac{1}{\kappa^2}\nabla\cdot(\rho\nabla\varphi) - \rho\theta_\lambda\,\varphi(\varphi^2-1) - \rho(\theta + \lambda p + \mathbf{v}_s^2)\varphi \tag{17}$$

$$\nabla\cdot(\rho\varphi^2\mathbf{v}_s) = -\tau\kappa^2\rho\varphi^2\phi_s \tag{18}$$

$$\partial_t\mathbf{v}_s = -\nabla\phi_s - \nabla\times\mathbf{v}_n - \rho\varphi^2\mathbf{v}_s + \nabla\theta \tag{19}$$

$$\begin{aligned}\rho\partial_t\mathbf{v}_n =& -\nabla\times\nabla\times\mathbf{v}_n + \nu\nabla(\nabla\cdot\mathbf{v}_n) - \nabla p\\ & -\nabla\times(\rho\varphi^2\mathbf{v}_s) + \rho\mathbf{g},\end{aligned} \tag{20}$$

$$\nabla\cdot\mathbf{v}_n = \lambda\rho\varphi\varphi_t \tag{21}$$

$$\rho_t = -\nabla\cdot(\rho\mathbf{v}_n + \rho\varphi^2\mathbf{v}_s) \tag{22}$$

$$\begin{aligned}\rho e_0'(\theta)\theta_t =& \tau\rho\varphi_t^2 + \rho\theta\varphi\varphi_t + |\partial_t\mathbf{v}_s + \nabla\phi_s - \nabla\theta|^2 + \nu|\nabla\cdot\mathbf{v}_n|^2\\ & + \tau\kappa^2\rho\varphi^2\phi_s^2 + \nabla\cdot(k_0(\theta)\nabla\theta) + \nabla\cdot(\rho\varphi^2\mathbf{v}_s)\theta + \rho r.\end{aligned} \tag{23}$$

in the unknowns $\varphi, \mathbf{v}_s, \phi_s, \mathbf{v}_n, p, \rho, \theta$.

In order to emphasize the analogy of our model with the classical Ginzburg-Landau model of superconductivity, following [12], we introduce the transformation:

$$(\varphi, \mathbf{v}_s, \phi_s) \quad \longleftrightarrow \quad (\psi, \mathbf{A}, \phi)$$

where

$$\varphi = \psi e^{-i\chi}, \quad \mathbf{v}_s = \mathbf{A} - \frac{1}{\kappa}\nabla\chi, \quad \phi_s = \phi + \frac{1}{\kappa}\chi_t,$$

χ is an arbitrary scalar function and i denotes the imaginary unit.

Our aim is to write system (17)-(23) by means of the variables $(\psi, \mathbf{A}, \phi)$ instead of $(\varphi, \mathbf{v}_s, \phi_s)$. Multiplying equation (17) by $e^{i\chi}$, we obtain

$$\tau\rho\psi_t = \left(i\tau\rho\varphi\chi_t + \frac{1}{\kappa^2}\rho\Delta\varphi + \frac{1}{\kappa^2}\nabla\rho\cdot\nabla\varphi\right)e^{i\chi} - \rho\psi[\theta_\lambda(|\psi|^2-1)+\theta+\lambda p]$$
$$-\rho\psi\left(|\mathbf{A}|^2 - \frac{2}{\kappa}\mathbf{A}\cdot\nabla\chi + \frac{1}{\kappa^2}|\nabla\chi|^2\right). \tag{24}$$

Dividing (18) by $\kappa\varphi$ and substituting the expressions of $\mathbf{v}_s$ and ϕ_s, we deduce

$$\tau\rho\varphi\chi_t = -\frac{1}{\kappa}\varphi\mathbf{A}\cdot\nabla\rho + \frac{1}{\kappa^2}\varphi\nabla\rho\cdot\nabla\chi - \frac{2}{\kappa}\rho\mathbf{A}\cdot\nabla\varphi + \frac{2}{\kappa^2}\rho\nabla\varphi\cdot\nabla\chi$$
$$-\frac{1}{\kappa}\rho\varphi\nabla\cdot\mathbf{A} + \frac{1}{\kappa^2}\rho\varphi\Delta\chi - \tau\kappa\rho\varphi\phi. \tag{25}$$

We substitute equation (25) into (24). Moreover, the identities

$$\nabla\psi = (\nabla\varphi + i\varphi\nabla\chi)e^{i\chi},$$
$$\Delta\psi = (\Delta\varphi + 2i\nabla\varphi\cdot\nabla\chi - \varphi|\nabla\chi|^2 + i\varphi\Delta\chi)e^{i\chi},$$

lead to

$$\tau\rho\psi_t = \frac{1}{\kappa^2}\rho\Delta\psi + \frac{1}{\kappa^2}\nabla\psi\cdot\nabla\rho - \frac{i}{\kappa}\psi\mathbf{A}\cdot\nabla\rho - \frac{2i}{\kappa}\rho\mathbf{A}\cdot\nabla\psi$$
$$-\frac{i}{\kappa}\rho\psi\nabla\cdot\mathbf{A} - i\tau\kappa\rho\psi\phi - \rho\psi[\theta_\lambda(|\psi|^2-1)+\theta+\lambda p] - \rho\psi|\mathbf{A}|^2.$$

In addition, the following relations can be easily proved

$$\partial_t\mathbf{v}_s + \nabla\phi_s = \mathbf{A}_t + \nabla\phi$$
$$\varphi^2\mathbf{v}_s = -|\psi|^2\mathbf{A} + \frac{i}{2\kappa}(\psi\nabla\psi^* - \psi^*\nabla\psi)$$
$$\varphi\varphi_t = \frac{1}{2}(\psi\psi_t^* + \psi^*\psi_t)$$
$$\varphi_t^2 + \kappa^2\varphi^2\phi_s^2 = |\psi_t|^2 + \kappa^2\varphi^2\phi^2 + 2\kappa\phi\varphi^2\chi_t$$
$$= |\psi_t|^2 + \kappa^2|\psi|^2\phi^2 - i\kappa\phi(\psi_t\psi^* - \psi\psi_t^*),$$

DOI: 10.1685/journal.caim.425

where ψ^* denotes the conjugate of ψ. Thus, equations (17) and (23) read

$$\begin{aligned}
\tau\rho\psi_t &= \frac{1}{\kappa^2}\rho\Delta\psi - \frac{i}{\kappa}\psi\mathbf{A}\cdot\nabla\rho + \frac{1}{\kappa^2}\nabla\psi\cdot\nabla\rho - \frac{2i}{\kappa}\rho\mathbf{A}\cdot\nabla\psi \\
&\quad -\frac{i}{\kappa}\rho\psi\nabla\cdot\mathbf{A} - i\tau\kappa\rho\psi\phi - \rho\psi[\theta_\lambda(|\psi|^2-1)+\theta+\lambda p] - \rho\psi|\mathbf{A}|^2 \\
\mathbf{A}_t &= -\nabla\phi - \nabla\times\mathbf{v}_n + |\psi|^2\mathbf{A} - \frac{i}{2\kappa}(\psi\nabla\psi^* - \psi^*\nabla\psi) + \nabla\theta \\
\rho\partial_t\mathbf{v}_n &= -\nabla\times\nabla\times\mathbf{v}_n + \nu\nabla(\nabla\cdot\mathbf{v}_n) - \nabla p \\
&\quad -\nabla\times\left[-\rho|\psi|^2\mathbf{A} + \frac{i}{2\kappa}\rho(\psi\nabla\psi^* - \psi^*\nabla\psi)\right] + \rho\mathbf{g} \\
\nabla\cdot\mathbf{v}_n &= \frac{\lambda}{2}\rho(\psi\psi_t^* + \psi^*\psi_t) \\
\rho_t &= -\nabla\cdot\left[\rho\mathbf{v}_n - \rho|\psi|^2\mathbf{A} + \frac{i}{2\kappa}\rho(\psi\nabla\psi^* - \psi^*\nabla\psi)\right] \\
\rho e_0'(\theta)\theta_t &= \frac{1}{2}\rho\theta(\psi\psi_t^* + \psi^*\psi_t) + \tau\rho|\psi_t|^2 + \tau\rho\kappa^2|\psi|^2\phi^2 - i\tau\kappa\rho\phi(\psi_t\psi^* - \psi\psi_t^*) \\
&\quad +|\mathbf{A}_t + \nabla\phi - \nabla\theta|^2 + \nu|\nabla\cdot\mathbf{v}_n|^2 + \nabla\cdot[k_0(\theta)\nabla\theta] \\
&\quad +\nabla\cdot\left[-\rho|\psi|^2\mathbf{A} + \frac{i}{2\kappa}\rho(\psi\nabla\psi^* - \psi^*\nabla\psi)\right]\theta + \rho r.
\end{aligned}$$

The previous equations are independent of χ, which can be chosen arbitrarily. Therefore this formulation allows a more direct comparison with the Ginzburg-Landau model of superconductivity, where the choice of the variables $(\psi, \mathbf{A}, \phi)$ is crucial in order to prove well-posedness results for the differential system and the long-time behaviour of the solution [13,14].

REFERENCES

1. L. Tisza, Transport phenomena in He II, *Nature*, vol. 141, p. 913., 1938.
2. L. Landau, On the theory of superfluidity of helium II, *Journal of Physics USSR*, vol. 5, pp. 71–77., 1941.
3. M. Fabrizio, A Ginzburg-Landau model for the phase transition in helium II, *Z. Angew. Math. Phys.*, vol. 61, pp. 329–340, 2010.
4. K. Mendelssohn in *Liquid helium*, Handbuch Physik, Flugge S, p. 370461, Springer, 1956.
5. D. Tilley and J. Tilley, *Superfluidity and superconductivity.* Graduate student series in physics 138, 1990.
6. M. Fabrizio, Ginzburg-Landau equations and first and second order phase transitions, *Internat. J. Engrg. Sci.*, vol. 44, pp. 529–539, 2006.
7. M. Frémond, *Non-smooth Thermomechanics.* Springer, 2002.

8. V. Ginzburg and L. Landau, On the theory of superconductivity, *Zh. Eksp. Teor. Fiz.*, vol. 20, pp. 1064–1082., 1950.
9. M. Mongioví, Nonlinear extended thermodynamics of a non-viscous fluid in the presence of heat flux, *J. Non-Equilib. Thermodyn*, vol. 25, pp. 31–47, 2000.
10. M. Mongioví, Extended irreversible thermodynamics of liquid helium II , *Phys. Rev. B*, vol. 48, pp. 6276–6283, 1993.
11. D. Jou, G. Lebon, and M. Mongioví, Second sound, superfluid turbulence, and intermittent effects in liquid helium II , *Phys. Rev. B*, vol. 66, pp. 224509–1 – 224509–9, 2002.
12. V. Berti, M. Fabrizio, and C. Giorgi, Gauge invariance and asymptotic behavior for the Ginzburg-Landau equations of superconductivity, *J. Math. Anal. Appl.*, vol. 329, pp. 357–375, 2007.
13. V. Berti and M. Fabrizio, Existence and uniqueness for a mathematical model in superfluidity, *Math. Meth. Appl. Sci.*, vol. 31, pp. 1441–1459, 2008.
14. A. Berti, V. Berti, and I. Bochicchio, Global and exponential attractors for a Ginzburg-Landau model of superfluidity, *Discret. Contin. Dyn. Syst. Ser. S*, vol. 4, no. 2, pp. 247–271, 2011.