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# Approximated solutions to a 3D-packing MIP model by a non-linear approach

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## Abstract

This article extends previous works focused on a Mixed Integer Programming (MIP) based heuristic approach, aimed at solving non-standard three-dimensional problems with additional conditions. The paper considers a Mixed Integer Non-linear (MINLP) reformulation of the general MIP model, to improve the former heuristic, based on linear relaxation. Insights on the modeling aspects and experimental outcomes are provided, with cues on expected future research. This paper refers to the presentation delivered at the SIMAI Conference 2012 [1].

*Keywords:* `three-dimensional packing, MIP/MINLP models, linear/non-linear approximations, heuristics.`

*AMS subject classification:* `05B40, 90C11, 90C30, 90C59, 90C90.`

## 1. Introduction.

This work arises from ongoing research within the space engineering framework, in support to the cargo accommodation of space vehicles and modules. Due to the limitation of the space available and the remarkable number of items to accommodate, the objective of this activity is that of maximizing the loaded cargo (in terms of volume or mass), in compliance with the given accommodation rules. Very complex geometrical aspects have to be taken into account, in addition to quite demanding balancing conditions, deriving from tight attitude control specifications.

Small items can often be modeled as parallelepipeds (cuboids), even if their actual shapes are not exactly so. Nonetheless, this approximation is no longer sustainable when large items are involved and, therefore, clusters of mutually orthogonal parallelepipeds, such as *tetris*-like items, or composite prisms, are instead more suitable. Similar considerations may be taken account of, moreover, as far as the container shape is concerned.

A number of specialist works in the Operations Research context deal with Mixed Integer Programming (MIP) formulations of packing problems (e.g. [2–4]). The subject discussed herewith extends previous works aimed at solving non-standard 3D-packing problems, involving *tetris*-like items, convex domains (with possible holes and separation planes) and presence of additional conditions, such as balancing [5,6]. In these works a MIP-based heuristic has been proposed to efficiently tackle real-world instances. A first step consists of an LP-relaxation of the non-intersection constraints between items and the adoption of an ad hoc objective

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function aimed at providing the further steps of the process with an approximate solution to the original problem, expressed in term of a feasibility one.

Vast literature is available nowadays on Mixed Integer Non-linear Programming and Global Optimization (MINLP, GO, [7–28]) encouraging their application to several classes of challenging optimization problems, including hard packing issues [29–38].

The $\phi$ function concept has been introduced by Stoyan [33,39] to solve arduous irregular nesting problems, involving complex objects and, following a different point of view, Birgin and Martinez [40] and Cassioli and Locatelli [41] investigate a non-linear-based approach for packing rectangles inside a convex region.

A novel non-linear approach is outlined in this article. It is aimed at improving the approximate initial LP-relaxed solution of the MIP-based heuristic previously introduced [5,6]. The approach proposed in the following stresses the linear structure of the packing MIP model. It is addressed in particular to standard MINLP solvers with special algorithmic features, aimed at exploiting the presence of linear constraint sets.

Section 2 briefly reviews the previous MIP-based approach. Section 3 discusses the non-linear re-formulation of its model, aimed at improving the previous MIP-based heuristic, focusing on the *non-intersection* constraints. Section 4 reports an extensive experimental analysis, carried out in different computational environments.

## 2. MIP formulation and LP-relaxed approximation.

The general MIP model, together with its LP-relaxed approximation, is outlined in this section referring to the (three-dimensional) loading problem taken into account in previous works [5,6]. These are focused on the orthogonal packing of *tetris*-like items within a convex domain (with possible separation planes and forbidden zones) and subject to further additional conditions, such as balancing. Nevertheless, for the sake of simplicity, the discussion herewith will consider only items consisting of single parallelepipeds to load inside a convex domain, neglecting any possible further conditions (e.g. separation planes, forbidden zones, balancing). In the following, the only packing rules to consider are therefore:

- each item (parallelepiped) side has to be parallel to an axis of a prefixed orthonormal reference frame (*orthogonality conditions*);
- each item has to be contained within the given domain $D$ (*domain conditions*);
- items cannot overlap (*non-intersection conditions*).

In the previous works, the objective function consisted of maximizing either the loaded volume or mass. As pointed out there, the MIP formulation of the packing issue stated above gives rise, when dealing with real-world instances, to extremely hard to solve optimization problems. This is caused in particular by the *non-intersections* conditions, consisting of *big-M* constraints (given a set of $n$ items, $\mathcal{O}(3n(n-1))$ *big-M* constraints with their relative binary variables have to be generated).

In the previous works a heuristic approach was proposed to efficiently solve the above packing problem (or even more complex versions of it). The first step of the process considers the problem in terms of feasibility (i.e. all the given items have to be picked) and performs a linear relaxation of the *non-intersection* constraints, in order to obtain an approximate solution (with possible intersections between items). An ad hoc (linear) target function, aimed at *minimizing* the intersection between items has been introduced. The approximated solution thus obtained is utilized to generate an *abstract configuration*, that is, a set of relative positions (each one for each couple of items) that would be feasible in any unbounded domain. The *abstract configuration* is then imposed to the original problem, eliminating a number of items, if necessary. The loaded volume or mass is maximized, combining, by a recursive process, *exchange* and *hole-filling* techniques.

The feasibility model, on which the first step of the heuristic process is based, is briefly outlined herewith, referring the reader to previous works for more details. The constraints relative to the basic packing rules (*orthogonality*, *domain* and *non-intersection conditions*) are reported here below.

(1) $\forall\alpha, \forall i \qquad \sum_{\beta=1}^{3} \delta_{\alpha\beta i} = 1$

(2) $\forall\beta, \forall i \qquad \sum_{\alpha=1}^{3} \delta_{\alpha\beta i} = 1$

(3) $\forall\beta, \forall i, \forall\eta \qquad w_{\beta i} \pm \frac{1}{2}\sum_{\alpha=1}^{3} L_{\alpha i}\delta_{\alpha\beta i} = \sum_{\gamma=1}^{r} \psi_{\gamma\eta i} V_{\beta\gamma}$

(4) $\forall i, \forall\eta \qquad \sum_{\gamma=1}^{r} \psi_{\gamma\eta i} = 1$

(5a) $\forall\beta, \forall i, \forall j, i < j \qquad w_{\beta i} - w_{\beta j} \geq \frac{1}{2}\sum_{\alpha=1}^{3}(L_{\alpha i}\delta_{\alpha\beta i} + L_{\alpha j}\delta_{\alpha\beta j}) + d^{+}_{\beta ij} - D_{\beta}$

(5b) $\forall\beta, \forall i, \forall j, i < j \qquad w_{\beta j} - w_{\beta i} \geq \frac{1}{2}\sum_{\alpha=1}^{3}(L_{\alpha i}\delta_{\alpha\beta i} + L_{\alpha j}\delta_{\alpha\beta j}) + d^{-}_{\beta ij} - D_{\beta}$

(6a) $\forall\beta, \forall i, \forall j, i < j \qquad d^{+}_{\beta ij} \geq \sigma^{+}_{\beta ij} D_{\beta}$

(6b) $\forall\beta, \forall i, \forall j, i < j \qquad d^{-}_{\beta ij} \geq \sigma^{-}_{\beta ij} D_{\beta}$

(7) $\forall i, \forall j, i < j \qquad \sum_{\beta=1}^{3}(\sigma^{+}_{\beta ij} + \sigma^{-}_{\beta ij}) = 1$

where:

- given the reference frame $O, w_\beta, \beta \in \{1, 2, 3\}$, for each item $i \in \{1, 2, \ldots, n\}$ of sides $L_{\alpha i}$, $\alpha \in \{1, 2, 3\}$, the variables $\delta_{\alpha\beta i} \in \{0, 1\}$ have been introduced with the condition $\delta_{\alpha\beta i} = 1$ if $L_{\alpha i}$ is parallel to the $w_\beta$ axis and $\delta_{\alpha\beta i} = 0$ otherwise;
- $V_{\beta\gamma}$ are the vertices of the convex domain $D$, $\psi_{\gamma\eta i}$ are non negative variables;
- $D_\beta$ are the sides of the parallelepiped of minimum volume enveloping $D$;
- $d^{+}_{\beta ij}$, $d^{-}_{\beta ij} \in [0, D]$ and $\sigma^{+}_{\beta ij}$, $\sigma^{-}_{\beta ij} \in \{0, 1\}$.

Constraints (1) and (2) represent the *orthogonality* conditions; (3) and (4) the *domain* ones; (5), (6), (7) the *non-intersection* ones.

The goal of the first step of the heuristic process is to obtain a good approximate initial solution. The MIP model described above is adopted and the integrality conditions on the $\sigma^{\pm}_{\beta ij}$ variables are dropped. In this case constraints (6) and (7) can be replaced by

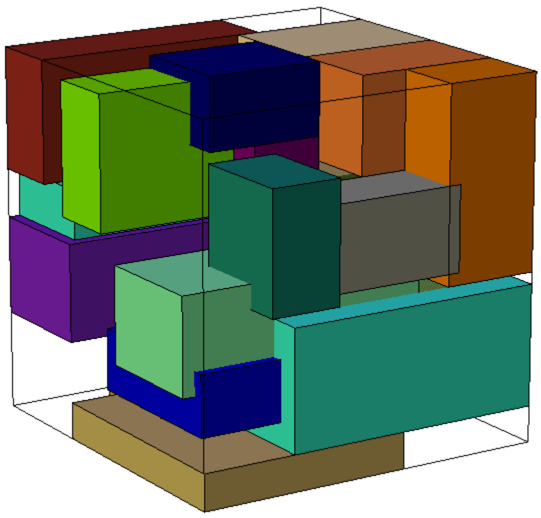

Figure 1: LP-relaxed approximate solution.

$$\sum_{\beta=1}^{3}\left(\frac{d_{\beta ij}^{+}}{D_\beta}+\frac{d_{\beta ij}^{-}}{D_\beta}\right)=1.$$

The ad hoc objective function (aimed at minimizing the intersection between items) is the following:

$$\max \sum_{\beta,i<j}(d_{\beta ij}^{+}+d_{\beta ij}^{-}). \tag{8}$$

Figure 1 depicts an approximate solution (with possible intersections between items) obtained by the linear (LP) relaxation described above.

## 3. MINLP-based approach.

### 3.1. *Non-linear formulation of non-intersection constraints.*

We introduce here a possible non-linear reformulation [42,43] of the *non-intersection* constraints reported in Section 2. Keeping the same meaning of the symbols previously adopted, it is straightforward to prove that the following non-linear constraints are equivalent to (5), (6) and (7):

$$\forall\beta,\forall i,\forall j, i<j \quad (w_{\beta i}-w_{\beta j})^2-\left[\frac{1}{2}\sum_{\alpha=1}^{3}(L_{\alpha i}\delta_{\alpha\beta i}+L_{\alpha j}\delta_{\alpha\beta j})\right]^2=s_{\beta ij}-r_{\beta ij} \tag{9}$$

$$\forall i,\forall j, i<j \quad \prod_{\beta=1}^{3} r_{\beta ij}=0 \tag{10}$$

where $s_{\beta ij} \in [0, D_\beta^2]$ and $r_{\beta ij} \in [0, D_\beta^2]$.

Indeed for each couple $i, j (i < j)$ equations (10) guarantee that the terms $r_{\beta ij}$ are zero for at least one $\beta$ and equation (9), corresponding to such $\beta$, is equivalent to the *non-intersection* condition:

$$|w_{\beta i} - w_{\beta j}| \geq \frac{1}{2} \sum_{\alpha=1}^{3} (L_{\alpha i}\delta_{\alpha\beta i} + L_{\alpha j}\delta_{\alpha\beta j}).$$

Constraints (5) and (6) correspond therefore to equations (9) and vice versa, while equation (7) corresponds to (10) and vice versa.

3.2. *Non-linear approximation.*

In the following, we reformulate the feasibility problem reported in Section 2, on the basis of what discussed in Section 3.1, by introducing a new ad hoc non-linear target function aimed at minimizing the intersection between items. As the *non-intersection* constraints (9) and (10) are most likely quite hard to tackle, they are expressed as penalty terms in the new objective function. All remaining constraints, on the contrary, as they are linear (MIP), are maintained as such.

Since obtaining a global optimal solution to the above MINLP problem is, in general, an extremely hard task, we do not expect to solve it tout court. A joint use of the LP-relaxation and the MINLP approach, within the heuristic process outlined in Section 2, is nevertheless expected to be of advantage. The LP-relaxed solution can be exploited by the MINLP process, looking into sub-optimal solutions, to further reduce the overall intersection between items.

To reformulate the original feasibility (MIP) problem, we shall consider the following MINLP one:

$$(11)\quad \min\left\{ \sum_{\beta,i<j} \left\{ (w_{\beta i} - w_{\beta j})^2 - \left[\frac{1}{2}\sum_{\alpha=1}^{3}(L_{\alpha i}\delta_{\alpha\beta i} + L_{\alpha j}\delta_{\alpha\beta j})\right]^2 - s_{\beta ij} + r_{\beta ij} \right\}^2 + \sum_{i<j}\left(\prod_{\beta} r_{\beta ij}\right)\right\}$$

subject to the constraints (1), (2), (3), (4) reported in Section 2.

It is immediately seen that the objective function is non-negative and that any zero-global optimal solution exists if and only if constraints (1), (2), (3), (4), (5), (6), (7) of the MIP problem reported in Section 2 delimit a feasible region.

The above target function thus minimizes the *non-intersection* between items, but, differently from target function (8), its global optima guarantee an ultimate (non approximate) solution to the feasibility problem of Section 3.1 and its adoption is therefore *a priori* preferable to the other. The MINLP model described above maintains all the MIP constraints present in the original model, so that, even its sub-optimal solutions guarantee to satisfy all *orhogonality* and *domain* conditions. Further MIP constraints could be directly added to the MINLP model in order to contemplate additional conditions, such as balancing. The extension to include *tetris*-like items would also be straightforward [5,6].

3.3. *Alternative non-linear formulation and approximation.*

It is also possible to provide other *non-linear* formulations of the packing problem, focusing our attention on the *non-intersection* constraints of the MIP and LP-relaxed models.

We start analyzing what can happen to inequalities (5a) and (5b), according to different item relative positions. We observe that, usually, if two items overlap along a particular axis, one of the variables $d^+_{\beta ij}$ and $d^-_{\beta ij} \in [0, D]$, associated to that axis, is positive and less than $D_\beta$, while the other is set to zero. In the opposite case, when two items do not overlap, the positive variable is exactly equal to $D_\beta$, its upper bound, while the other remains equal to zero. Therefore, to limit item overlapping along a particular axis, we ask that one of the two variables associated to this axis is as big as possible. Moreover, as the other one tends to be zero, it is equivalent to require that the difference between these variables is as big as possible. Since we do not know *a priori* the item relative positions, we need to distinguish between the case in which item $i$ precedes item $j$ and the converse. Using the second power applied to this difference, we can include both cases in only one expression, obtaining the following overall *non-linear* objective function:

$$\max \sum_{\beta, i<j} (d^+_{\beta ij} - d^-_{\beta ij})^2. \tag{12}$$

Therefore, using (12), together with *non-intersection* constraints (5a) and (5b), we define a new non-linear model quite different from the preceding one, in particular for some critical aspects.

The first concerns the non-overlapping requirement. Indeed it would be sufficient that the *non-intersection* conditions hold just for one axis. With the non-linear objective function (12), instead, the solver is addressed to arranging items according to a "stronger" objective and, as a consequence, the problem becomes harder to solve.

Secondly, in the first non-linear model formulation, the objective function (11) is non-negative and needs to be minimized, so we are sure of finding the global optimal solution when it reaches its zeros. On the contrary, using objective function (12), to be maximized, we do not have *a priori* known value corresponding to the global optimal solution. Therefore, we are not able to evaluate how close we are to an optimal solution.

## 4. Experimental analysis.

A remarkable effort has been carried out to perform a dedicated experimental analysis, currently ongoing [44]. A survey of the relevant outcomes is reported in this section, providing insights on possible enhancements, both from the computational and modeling point of views, to promote further research.

4.1. *Basic context.*

Having defined the non-linear formulations for the 3D-packing problem, we start to analyze their behavior with concrete problems. We therefore create a group of tests with differently sized domains and with a different number of items to accommodate.

As far as the solver is concerned, in order to perform the non-linear analysis we used the Lipschitz Global Optimizer LGO developed by Pintér [45]. It is important to underline that a focal starting point was about the searching method used in our analysis. Even if LGO could operate the global optimum search according to four different methods (Branch & Bound, Global Adaptive Random Search, Multi-Start Random Search and a Local Search), we set LGO on the Multi-Start Random Search.

The original code was thus adapted to generate instances for both models, starting from the same approximate initial LP-relaxed (sub-optimal) solution, with the aim of obtaining comparable results in terms of three parameters: objective function value, execution time and overlapping volume.

Since the goal of the non-linear model is only to improve the approximate sub-optimal solution that loads all items allowing overlapping, we are not interested in finding the optimal solution without intersection at this step, but only in obtaining a better *abstract configuration* that could help the following phases to reach the final solution faster. If a global optimum is found, we are sure that all items are loaded without overlapping. Otherwise, as the non-linear objective function is addressed to reducing item overlapping, we hope for a better item disposition in which the intersection between items is reduced.

4.2. *Test analysis.*

We will now present a brief report [44] to compare the behavior of the two non-linear models, focusing our attention on the three parameters underlined in the previous section. To make the model identification easier, in the following we will refer to the model with objective function (11) and (12) as *r&s model* and *d model* respectively.

4.2.1. *Objective function value.*

Since the two objective functions (11) and (12) work in different ways, one minimizing to zero, while the other maximizing without any *a priori* known global optimum, we cannot directly compare the results according to this criterion. Therefore, we will firstly analyze the non-linear solver behavior with the *r&s model*, and secondly with the *d* one.

With reference to the *r&s model* formulation (see Figure 2), we observed that LGO (blue line) is normally able to reduce the objective function value even if, as predictable, the reduction decreases when the number of items to load is incremented. However, treating a limited number of items, LGO can find the zero of (11) quickly. Therefore, in terms of improvement percentage (red columns), it is quite clear, as is obvious, that the *r&s model* has a decreasing trend according to the increasing number of items to load. Starting from “easy” tests, in which the global optimal solution is usually found, we can notice that LGO is always able to reduce the objective function (42% of average reduction) until 10%-20% of improvement for “harder” tests.

Analyzing the *d model* formulation (Figure 3), an overall and constant slight objective function growth results evident (yellow line vs. the black one). And this holds even if the corresponding percentage of improvement (red columns) decreases as the number of items to accommodate increases (as happened with the *r&s model*). In particular, focusing our attention on the percentage of improvement, we can point out two different trends. As long as the number of items to accommodate is lower or equal to ten, there is an average improvement of 40%, whereas exceeding this value, this percentage falls to 10%, sometimes with no improvement at all.

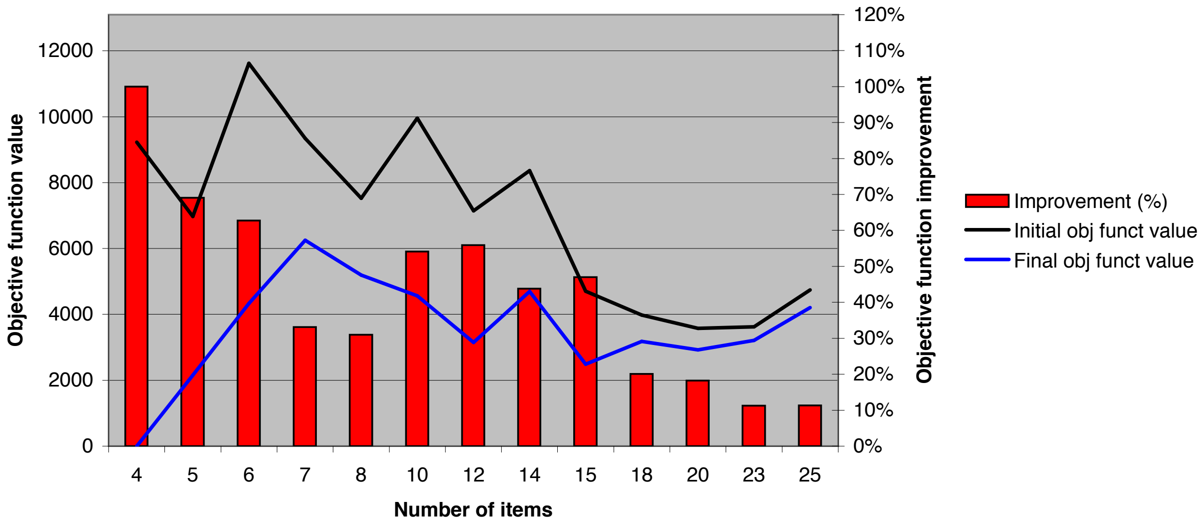


Figure 2: *r&s model* objective function value.

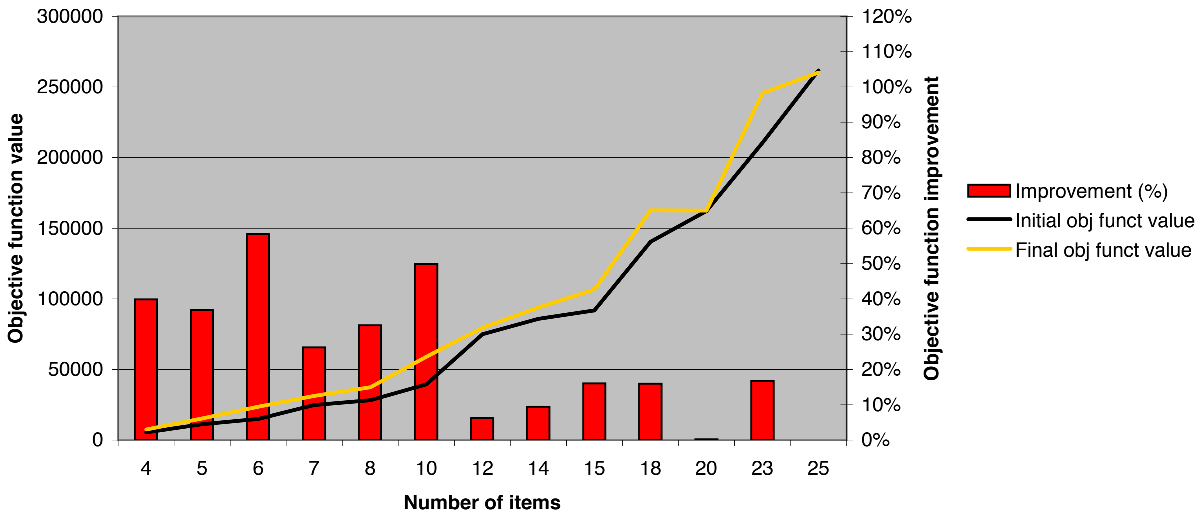


Figure 3: *d model* objective function value.

4.2.2. *Execution time.*

Before discussing the execution time, we need to underline that both models were tested with a time limit equal to an hour, never reached in either case.

As gathered, LGO for both models increases the execution time required to find the solution as the number of items grows. Comparing the two models from the computational effort pointy of view, Figure 4 highlights that the *d model* outperforms the *r&s* one.

4.2.3. *Overlapping volume.*

The overlapping volume is a significant parameter to analyze the validity of the solutions found, thus it could be the most important feature to consider for choosing a specific model.

Observing Figure 5, it is evident that the *d model* (yellow columns) has an unexpected

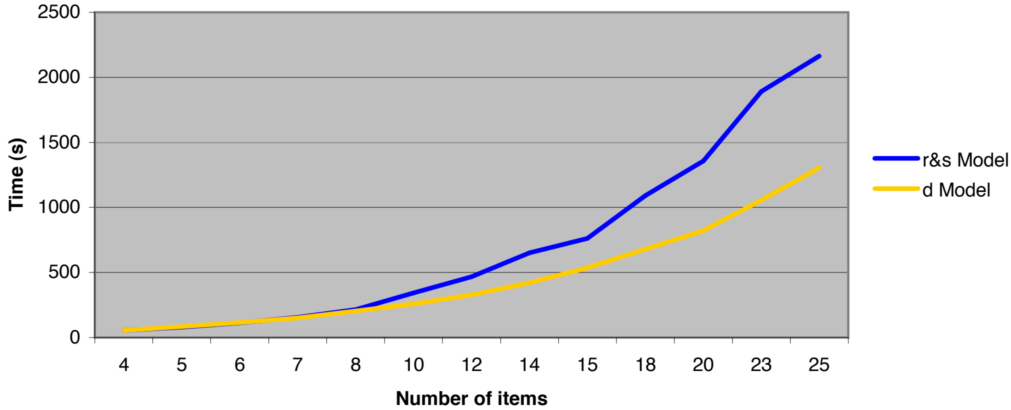


Figure 4: Execution time.

behavior since, for "medium-hard" tests, the solution proposed has a final overlapping volume higher than the initial one (black columns). This fact becomes evident with this model, but in general it is not linked to the particular model selected, since it also happened on other tests analyzed with the *r&s* one. The reason why the final overlapping volume sometimes exceeds the initial one is just because the overlapping volume is not directly expressed in the objective function, but only linked to it.

On the other hand, data referring to the *r&s model* (blue columns) underline that this model is always able to reduce the final overlapping volume, even if this improvement decreases as the tests become "harder" to solve.

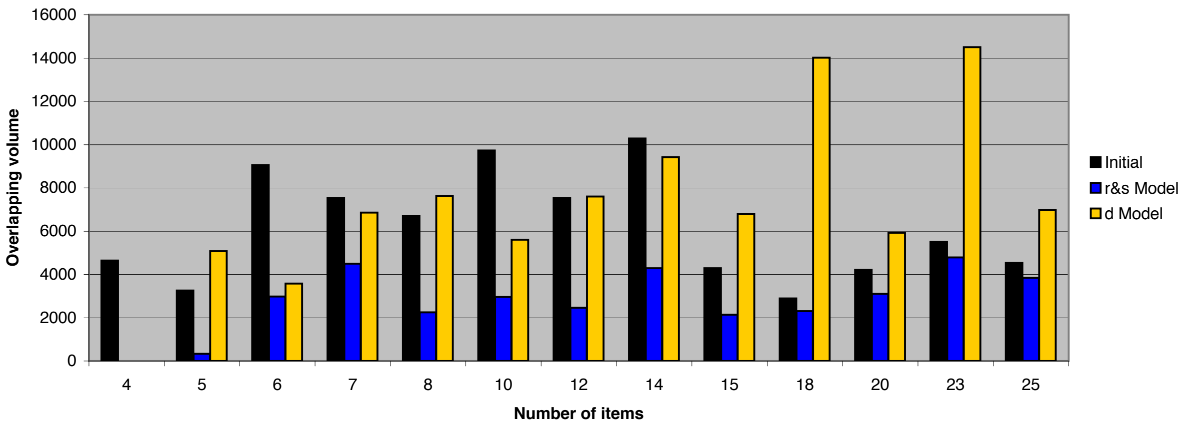


Figure 5: Overlapping volume.

4.2.4. *Observations.*

As an immediate consequence (although further consolidation is needed) of what outlined before, the best results are associated to the *r&s model*. Its formulation seems to be better addressed to reducing item overlapping as opposed to the *d model*, allowing the lowest final overlapping volumes, despite a slightly increased execution time. Referring to the non-linear solver, LGO seems to demonstrate great capability in analyzing and solving complex problems.

Therefore, a current analysis is focused on variations of the *r&s model*, in particular modifying the objective function (11) and following a strategy of penalization of some terms.

## 5. Conclusive remarks.

This article refers to previous works aimed at solving non-standard three-dimensional orthogonal packing problems with additional conditions. The approach followed in that context focuses on an MIP formulation and an MIP-based heuristic, introduced to efficiently solve hard real-world instances in practice.

The initial phase of this heuristic process addresses an LP-relaxation of the *non-intersection* constraints of the MIP model. The relaxed MIP model considers the original problem in terms of feasibility and an ad hoc (linear) objective function, aimed at minimizing the intersection between items, is adopted. The approximate solutions to this reformulated problem (with possible intersections) are taken as an initial step of the whole heuristic process.

In the present article, we have proposed a MINLP reformulations of the original MIP model, expressed in terms of feasibility. These reformulations are aimed at improving the approximate LP-relaxed solution of the MIP-based heuristic process.

The former MIP model has been reviewed in brief, to make the introduction of the proposed non-linear approach comprehensive. Non-linear reformulations of the *non-intersection* constraints have then been introduced in terms of objective functions. The *orthogonality* and *domain* conditions are, on the contrary, actually still treated as model constraints. An in-depth experimental analysis is currently ongoing. This article provides insights on the computational outcomes obtained to date.

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