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Flow of a viscous fluid through a porous medium when the thermal conductivity and the viscosity depend on the temperature and the pressure

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Abstract

Using the Hadamard-Caccioppoli global invertibility theorem we prove the existence and uniqueness of a problem of fluid flow in a porous medium assuming that the thermal conductivity and the viscosity depend on the temperature and on the pressure. A special case which can be reduced to the Riccati equation is also studied.

Keywords: Fluid flow, Porous media, Functional solution, Riccati equation.

AMS subject classification: 76D08, 35D99.

1. Introduction.

The thermal conductivity and the viscosity are crucial parameters in any diffusion processes and in many cases they depend sharply on the temperature. Less relevant is the dependence on the other basic thermodynamic quantity i.e. the pressure. However, this dependence exists as already noticed by C. Barus [1] in 1893. In recent years this dependence on the pressure has assumed considerable importance in engineering models [2] in relation to the new technique for extracting oil from rocks known as "hydrofracking". Section 4 of this paper deals with a problem of fluid flow in a porous medium assuming that the thermal conductivity and the viscosity depend on both the temperature and the pressure. A special case of this problem, treated with a two-point problem for the Riccati equation, is studied in Section 5. In Section 2, as a preliminary mathematical tool, we study the boundary value problem

$$\nabla \cdot (a(w,u)\nabla u + b(w,u)\nabla w) = 0 \quad \text{in } \Omega \tag{1}$$

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$$\nabla \cdot (K(w,u)\nabla w) = 0 \ \text{ in } \Omega \tag{2}$$

$$u = u_0 \ \text{ on } \Gamma_0,\ u = u_1 \ \text{ on } \Gamma_1 \tag{3}$$

$$w = w_0 \ \text{ on } \Gamma_0,\ w = w_1 \ \text{ on } \Gamma_1, \tag{4}$$

where Ω is a bounded subset of $\mathbf{R}^N$ with a regular boundary Γ consisting of two disjoint parts Γ_0 and Γ_1 both of class C^1 and u_0, u_1, w_0, w_1 are real numbers such that $w_1 > w_0$. We are in particular interested in the functional solutions of this problem according to the following

Definition 1.1. A classical solution of problem (1)-(4) is a functional solution if there exists a function $U(w) \in C^0([w_0,w_1]) \cap C^2((w_0,w_1))$ such that

$$u(\mathbf{x}) = U(w(\mathbf{x})),\ \mathbf{x} \in \bar{\Omega},\ \mathbf{x} = (x_1,..,x_N) \in \mathbf{R}^N.$$

Functional solutions permit to reformulate problem (1)-(4) as the following two-point problem for a first order ordinary differential equation depending on a numerical parameter

$$a(w,U)\frac{dU}{dw} + b(w,U) = \lambda K(w,U) \tag{5}$$

$$U(w_0) = u_0,\ U(w_1) = u_1. \tag{6}$$

In [3] and in [4] we proved that all the functional solutions of problem (1)-(4) can be computed if we can find the solutions of problem (5), (6) and the solution of the problem

$$\Delta z = 0 \ \text{ in } \Omega,\ z = 0 \ \text{ on } \Gamma_0,\ z = 1 \ \text{ on } \Gamma_1. \tag{7}$$

We quote in the lemma below this result and its proof for the reader's convenience. The example of non-uniqueness and the remarks on the uniqueness seems, however, to be new. Problem (5), (6) accounts for the nonlinearity of problem (1)-(4) and (7) for the corresponding geometrical aspects related to Ω and to its boundary. We can also characterize a class of problem of the form (1)-(4) for which all solutions are functional solutions. In Section 3 we present also a new proof of a theorem of K. Zawischa [5] and S. Takahaschi [6] based on that Lemma.

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2. The main elementary lemma

Lemma 2.1. *If* $a(w,u),\ b(w,u),\ K(w,u) \in C^0([w_0,w_1]\times \mathbf{R}^1)$ *and*

$$K(w,u) \geq k_0 > 0 \tag{8}$$

the functional solutions of problem (1)-(4) are in a one-to-one correspondence with the solutions of problem (5), (6).

Proof. Let $\mathcal{A}$ be the set of all solutions of the problem (5), (6) and denote by $\mathcal{B}$ the set of all solutions of the problem (1)-(4). We claim that

$$card\ \mathcal{A} = card\ \mathcal{B}.$$

Define $I : \mathcal{A} \to \mathcal{B}$ as follows. Let $(U(w),\lambda) \in \mathcal{A}$ and consider the nonlinear problem

$$\nabla \cdot (K(w,U(w))\nabla w) = 0 \ \text{ in } \Omega \tag{9}$$

$$w = w_0 \ \text{ on } \Gamma_0,\ w = w_1 \ \text{ on } \Gamma_1. \tag{10}$$

The problem (9), (10) has one and only one solution. For, let $v = F(w)$ with

$$F(w) = \int_{w_0}^{w} K(t,U(t))dt.$$

By (8) F maps one-to-one $[w_0,w_1]$ onto $[0,F(w_1)]$. In terms of $v(\mathbf{x})$ problem (9), (10) becomes

$$\Delta v = 0 \ \text{ in } \Omega,\ v = 0 \ \text{ on } \Gamma_0,\ v = F(w_1) \ \text{ on } \Gamma_1.$$

Hence $v(\mathbf{x}) = F(w_1)z(\mathbf{x})$ where $z(\mathbf{x})$ is the solution of (7). Thus we have $w(\mathbf{x}) = F^{-1}(F(w_1)z(\mathbf{x}))$. We claim that $(U(w(\mathbf{x})),w(\mathbf{x})) = I(U(w),\lambda) \in \mathcal{B}$. By (5) we have

$$\nabla \cdot (a(w,u)\nabla u + b(w,u)\nabla w) = \nabla \cdot \Big[\Big(a(w,U(w))\frac{dU}{dw} + b(w,U(w))\Big)\nabla w\Big] =$$

$$\lambda \nabla \cdot (K(w,U(w))\nabla w) = 0.$$

Moreover, $u = U(w_0) = u_0$ on Γ_0 and $u = U(w_1) = u_1$ on Γ_1. Hence the map I is well-defined. On the other hand, I is one-to-one. For, let $(U(w), \lambda) \in \mathcal{A}$, $(U'(w), \lambda') \in \mathcal{A}$ and $(u(\mathbf{x}), w(\mathbf{x})) = I(U(w), \lambda)$, $(u'(\mathbf{x}), w'(\mathbf{x})) = I(U'(w), \lambda')$, with $I(U(w), \lambda) = I(U'(w), \lambda')$. We have $u(\mathbf{x}) = u'(\mathbf{x})$, $w(\mathbf{x}) = w'(\mathbf{x})$ and $u(\mathbf{x}) = U(w(\mathbf{x}))$, $u'(\mathbf{x}) = U'(w(\mathbf{x}))$. Hence $U(w(\mathbf{x})) = U'(w(\mathbf{x}))$. On the other hand, $w_0 \leq w(\mathbf{x}) \leq w_1$ by the maximum principle. Thus

$$U(w) = U'(w)$$

and

$$a(w, U(w))\frac{dU}{dw} + b(w, U(w)) = \lambda K(w, U(w))$$

$$a(w, U(w))\frac{dU}{dw} + b(w, U(w)) = \lambda' K(w, U(w)).$$

By difference, taking into account (8) we have $\lambda = \lambda'$. Hence I is one-to-one. To prove that I is "onto" let $(u(\mathbf{x}), w(\mathbf{x})) \in \mathcal{B}$ and let $U(w)$ be the function such that $u(\mathbf{x}) = U(w(\mathbf{x}))$. We claim that $U(w)$ and the constant $\tilde{\lambda}$ determined below solve (5), (6). Define

$$\theta(w) = \int_{w_0}^{w} \Big[a(t, U(t))\frac{dU}{dw}(t) + b(t, U(t))\Big]dt \tag{11}$$

$$\zeta(w) = \int_{w_0}^{w} K(t, U(t))dt \tag{12}$$

and $\Theta(\mathbf{x}) = \theta(w(\mathbf{x}))$, $Z(\mathbf{x}) = \zeta(w(\mathbf{x}))$. We have $\nabla\Theta = a(w(\mathbf{x}), u(\mathbf{x}))\nabla u + b(w(\mathbf{x}), u(\mathbf{x}))\nabla w$ and, since $(u(\mathbf{x}), w(\mathbf{x}))$ solves (1)-(4),

$$\Delta\Theta = 0 \ \text{ in } \Omega, \quad \Theta = 0 \ \text{ on } \Gamma_0$$

$$\Theta = \theta(w_{|\Gamma_1}) = \theta(w_1) = \int_{w_0}^{w_1} \Big[a(t, U(t))\frac{dU}{dt}(t) + b(t, U(t))\Big]dt = \tilde{C} \ \text{ on } \Gamma_0.$$

Moreover $\nabla Z = K(w(\mathbf{x}), u(\mathbf{x}))\nabla w$. Hence, in view of (2),

$$\Delta Z = 0 \ \text{ in } \Omega$$

and

$$Z = \zeta(w_{|\Gamma_0}) = \zeta(w_0) = 0 \ \text{ on } \Gamma_0$$

$$Z = \zeta(w_{|\Gamma_1}) = \zeta(w_1) = \int_{w_0}^{w_1} K(t, U(t))dt = C \ \text{ on } \Gamma_1$$

with $C \neq 0$ by (8). If $z(\mathbf{x})$ is the solution of $\Delta z = 0$, $z = 0$ on Γ_0, $z = 1$ on Γ_1, we have $\Theta(\mathbf{x}) = \tilde{C}z(\mathbf{x})$ and $Z(\mathbf{x}) = Cz(\mathbf{x})$ and, on setting $\tilde{\lambda} = \frac{\tilde{C}}{C}$,

$$\Theta(\mathbf{x}) = \tilde{\lambda}Z(\mathbf{x}).$$

By (11) and (12) we have, for all $\mathbf{x} \in \Omega$,

$$\int_{w_0}^{w(\mathbf{x})} \left[a(t, U(t))\frac{dU}{dw}(t) + b(t, U(t))\right]dt = \tilde{\lambda}\int_{w_0}^{w(\mathbf{x})} K(t, U(t))dt.$$

Thus, for all $w \in [w_0, w_1]$,

$$\int_{w_0}^{w} \left[a(t, U(t))\frac{dU}{dw}(t) + b(t, U(t))\right]dt = \tilde{\lambda}\int_{w_0}^{w} K(t, U(t))dt.$$

Hence

$$a(w, U(w))\frac{dU}{dw}(w) + b(w, U(w)) = \tilde{\lambda}K(w, U(w))$$

and $I(U(w), \tilde{\lambda}) = (u(\mathbf{x}), w(\mathbf{x}))$ as required to prove that I is "onto". □

Remark 2.1. As an immediate consequence of the Lemma we can say that problem (1)-(4) has one and only one functional solution if the problem (5), (6) has one and only one solution and vice versa.

Remark 2.2. Under the sole assumption (8) problem (1)-(4) may have more than one solution as in the following example.

$$\nabla \cdot (uK(w, u)\nabla u + wK(w, u)\nabla w) = 0 \ \text{ in } \Omega \tag{13}$$

$$\nabla \cdot (K(w, u)\nabla w) = 0 \ \text{ in } \Omega \tag{14}$$

$$u = 0 \ \text{ on } \Gamma_0, \ u = 0 \ \text{ on } \Gamma_1 \tag{15}$$

$$w = 0 \ \text{ on } \Gamma_0, \ w = w_1 \ \text{ on } \Gamma_1. \tag{16}$$

In view of (8) the associated two-point problem is

$$U\frac{dU}{dw} + w = \lambda, \ U(0) = 0, \ U(w_1) = 0.$$

We have the two solutions:

$$U(w) = \sqrt{w_1 w - w^2}, \ \tilde{U}(w) = -\sqrt{w_1 w - w^2}.$$

Let, for $w \in [0, w_1]$,

$$G(w) = \int_0^w K(t, \sqrt{w_1 t - t^2}) \ dt, \ \tilde{G}(w) = \int_0^w K(t, -\sqrt{w_1 t - t^2}) \ dt$$

and let $v(\mathbf{x})$, $\tilde{v}(\mathbf{x})$ be respectively the solutions of

$$\Delta v = 0 \ \text{ in } \Omega, \ v = 0 \ \text{ on } \Gamma_0, \ v = G(w_1) \ \text{ on } \Gamma_1$$

$$\Delta \tilde{v} = 0 \ \text{ in } \Omega, \ \tilde{v} = 0 \ \text{ on } \Gamma_0, \ \tilde{v} = \tilde{G}(w_1) \ \text{ on } \Gamma_1.$$

Then

$$(w(\mathbf{x}), u(\mathbf{x})) = (G^{-1}(v(\mathbf{x})), \sqrt{w_1 G^{-1}(v(\mathbf{x})) - (G^{-1}(v(\mathbf{x})))^2})$$

$$(\tilde{w}(\mathbf{x}), \tilde{u}(\mathbf{x})) = (\tilde{G}^{-1}(\tilde{v}(\mathbf{x})), \sqrt{w_1 \tilde{G}^{-1}(\tilde{v}(\mathbf{x})) - (\tilde{G}^{-1}(\tilde{v}(\mathbf{x})))^2})$$

are the two functional solutions of (13)-(16).

Remark 2.3. The simplest nontrivial example of problem (1)-(4) is

$$\nabla \cdot (K(w, u)\nabla u) = 0 \ \text{ in } \Omega \tag{17}$$

$$\nabla \cdot (K(w, u)\nabla w) = 0 \ \text{ in } \Omega \tag{18}$$

$$u = 0 \ \text{ on } \Gamma_0, \ \ u = u_1 \ \text{ on } \Gamma_1 \tag{19}$$

$$w = 0 \quad \text{on } \Gamma_0, \quad w = w_1 \quad \text{on } \Gamma_1, \quad w_1 > 0. \tag{20}$$

The solution of the two-point problem associated with (17)-(20), assuming (8), is $U(w) = \frac{u_1}{w_1} w$. If $w(\mathbf{x})$ is the (unique) solution of the problem

$$\nabla \cdot \Big(K(w, \frac{u_1}{w_1} w) \nabla w \Big) = 0$$

$$w = 0 \quad \text{on } \Gamma_0, \ w = w_1 \quad \text{on } \Gamma_1$$

we have as the unique functional solution of problem (17)-(20)

$$(u(\mathbf{x}), w(\mathbf{x})) = \Big(\frac{u_1}{w_1} w(\mathbf{x}), w(\mathbf{x}) \Big). \tag{21}$$

We claim that, in this case (21) is the unique solution of (17)-(20) in general i.e. (17)-(20) has no other solutions in addition to (21). For, let $(u^*(\mathbf{x}), w^*(\mathbf{x}))$ be any solution of (17)-(20) and define

$$\zeta(\mathbf{x}) = u^*(\mathbf{x}) - \frac{u_1}{w_1} w^*(\mathbf{x}).$$

We have

$$\nabla \cdot (K(w^*, u^*) \nabla \zeta) = \nabla \cdot (K(w^*, u^*) \nabla u^*) - \frac{u_1}{w_1} \nabla \cdot (K(w^*, u^*) \nabla w^*) = 0 \tag{22}$$

$$\zeta = 0 \quad \text{on } \Gamma_0, \ \zeta = 0 \quad \text{on } \Gamma_1. \tag{23}$$

Multiplying (22) by ζ and integrating by parts over Ω we obtain

$$\int_\Omega K(w^*, u^*) |\nabla \zeta|^2 dx = 0. \tag{24}$$

Hence, in view of (8), (24) and (23), we have $\zeta(\mathbf{x}) = 0$. Thus $u^*(\mathbf{x}) = \frac{u_1}{w_1} w(\mathbf{x})$. Therefore $(u^*(\mathbf{x}), w^*(\mathbf{x}))$ is a functional solution, but in this case there is only one functional solution. Hence problem (17)-(20) has a unique solution. In this order of ideas we can prove that the problem

$$\nabla \cdot (a(u, w) \nabla u + b(u, w) \nabla w) = 0 \quad \text{in } \Omega \tag{25}$$

$$\nabla \cdot (K(u,w)\nabla w) = 0 \quad \text{in } \Omega \tag{26}$$

$$u = u_0 \quad \text{on } \Gamma_0, \;\; u = u_1 \quad \text{on } \Gamma_1 \tag{27}$$

$$w = w_0 \quad \text{on } \Gamma_0, \;\; w = w_1 \quad \text{on } \Gamma_1, \tag{28}$$

has a unique functional solution and there are no other solutions if

$$a(w,u) \geq a_0 > 0 \tag{29}$$

$$K(w,u) \geq K_0 > 0, \tag{30}$$

and we assume that the differential form

$$\frac{a(w,u)}{K(w,u)}du + \frac{b(w,u)}{K(w,u)}dw$$

has a primitive $\Phi(w,u) \in C^1([w_0,w_1] \times \mathbf{R}^1)$. For, if we define

$$\theta = \Phi(w,u)$$

we can restate problem (25)-(28) as follows

$$\nabla \cdot (K(w,u)\nabla \theta) = 0 \quad \text{in } \Omega \tag{31}$$

$$\nabla \cdot (K(w,u)\nabla w) = 0 \quad \text{in } \Omega \tag{32}$$

$$\theta = \theta_0 = \Phi(w_0,u_0) \quad \text{on } \Gamma_0, \;\; \theta = \theta_1 = \Phi(w_1,u_1) \quad \text{on } \Gamma_1$$

$$w = w_0 \quad \text{on } \Gamma_0, \;\; w = w_1 \quad \text{on } \Gamma_1.$$

On the other hand, in view of (29) and (30), $\theta = \Phi(w,u)$ can be globally solved with respect to u with inverse

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$$u = G(w, \theta).$$

We obtain from (31) and (32)

$$\nabla \cdot (K(w, G(w, \theta))\nabla\theta) = 0 \quad \text{in } \Omega \tag{33}$$

$$\nabla \cdot (K(w, G(w, \theta))\nabla w) = 0 \quad \text{in } \Omega \tag{34}$$

$$\theta = \theta_0 \quad \text{on } \Gamma_0, \;\; \theta = \theta_1 \quad \text{on } \Gamma_1 \tag{35}$$

$$w = w_0 \quad \text{on } \Gamma_0, \;\; w = w_1 \quad \text{on } \Gamma_1. \tag{36}$$

Redefining K the problem (33)-(36) has the same structure of problem (17)-(20) and we may conclude that problem (25)-(28), in this case, has one and only one solution.

3. On a theorem of K. Zawischa and S. Takahaschi.

G. Zawischa in [5] and S. Takahaschi in [6] proved the following

Theorem 3.1. *Let $K(w, U)$ be a continuous function defined in the rectangle $R = \{(w, U) \in \mathbf{R}^2;\; |w - \tilde{w}| < a,\; |U - \tilde{u}| < b\}$ such that*

$$k_1 \geq K(w, U) \geq k_0 > 0 \quad \textit{for all} \;\; (w, U) \in R \tag{37}$$

and satisfying a Lipschitz condition with respect to U i.e.

$$|K(w, U_1) - K(w, U_2)| \leq N|U_1 - U_2| \quad \textit{for all} \;\; (w, U) \in R. \tag{38}$$

Let (w_0, u_0), $(w_1, u_1) \in R$, $w_1 > w_0$, then there exists one and only one solution of the problem

$$\frac{dU}{dw} = \lambda K(w, U) \tag{39}$$

$$U(w_0) = u_0, \;\; U(w_1) = u_1. \tag{40}$$

This theorem has received many improvements and extensions. We quote in particular [7–9]. In this Section we show that the theorem of Zawischa and Takahaschi is consequence of Lemma 2.1 and of a result of M. Chipot [10] which we quote below for the problem at hand.

Theorem 3.2. *Let Ω be an open and bounded subset of $\mathbf{R}^N$ with boundary consisting of two disjoint regular parts Γ_0 and Γ_1. If $m(\mathbf{x}, v)$ is measurable with respect to $\mathbf{x} \in \mathbf{R}^N$ and satisfies*

$$m_1 \geq m(\mathbf{x}, v) \geq m_0 > 0$$

$$|m(\mathbf{x}, v) - m(\mathbf{x}, v')| \leq C|v - v'|$$

then the weak solution of the problem

$$v \in H^1(\Omega), \quad \nabla \cdot (m(\mathbf{x}, v)\nabla v) = 0 \quad \textit{in } \Omega, \;\; v = v_1 \quad \textit{on } \Gamma_0, \;\; v = v_1 \quad \textit{on } \Gamma_1,$$

where v_0 and $v_1 \in \mathbf{R}^1$, exists and is unique.

We apply Lemma 2.1 assuming $a(w, u) = 1$, $b(w, u) = 0$. Then problem (1)-(4) becomes

$$\Delta u = 0 \quad \text{in } \Omega, \;\; u = u_0 \quad \text{on } \Gamma_0, \;\; u = u_1 \quad \text{on } \Gamma_1 \tag{41}$$

$$\nabla \cdot (K(w, u)\nabla w) = 0 \quad \text{in } \Omega, \;\; w = w_0 \quad \text{on } \Gamma_0, \;\; w = w_1 \quad \text{on } \Gamma_1. \tag{42}$$

The two-point problem corresponding to (41) and (42) is precisely the problem studied by Zawischa i.e. (39) and (40). From (41) we obtain $u(\mathbf{x})$. Then we use Theorem 3.2 with $m(\mathbf{x}, v) = K(v, u(\mathbf{x}))$. By (37) and (38) all the assumptions of Theorem 3.2 are satisfied. Hence problem (41)-(42) has one and only one solution. Therefore the same is true for problem (39) and (40) by Lemma 2.1. This proves the Zawischa-Takahaschi theorem.

4. An application to the problem of "hydro-fracking".

For extracting oil from rocks liquids are pumped at very high pressure in the terrain with the technique of "hydro-fracking". Experimental evidence [2] suggests that in these situations the viscosity μ and the thermal conductivity κ depend not only from the temperature u but also from the pressure p. We assume that the velocity $\mathbf{v}$ of the fluid obeys the law of Darcy i.e.

$$\mathbf{v} = -s(u,p)\nabla p \tag{43}$$

where

$$s(u,p) = \frac{K}{\mu(u,p)}$$

where K is a positive constant. Assuming the fluid incompressible we have $\nabla \cdot \mathbf{v} = 0$. Hence

$$\nabla \cdot (s(u,p)\nabla p) = 0. \tag{44}$$

The density of the heat flow is given by $\mathbf{q} = -\kappa(u,p)\nabla u + \alpha \mathbf{v} u$, $\alpha > 0$, whereas the heat density generated by attrition in the medium is proportional to $-\mathbf{v} \cdot \nabla p$ ([11] page 647). This is equal, in view of (43), to $\beta s(u,p)|\nabla p|^2$, $\beta > 0$. Hence the energy equation reads $\nabla \cdot \mathbf{q} = \beta s(u,p)|\nabla p|^2$. On the other hand, by (44) we have $\nabla \cdot (s(u,p)p\nabla p) = s(u,p)|\nabla p|^2$, this permit to write the energy equation in full divergence form as follows

$$\nabla \cdot \big[\kappa(u,p)\nabla u + \alpha u s(u,p)\nabla p + \beta p s(u,p)\nabla p\big] = 0.$$

With a suitable choice of units we arrive at the problem

$$\nabla \cdot \big[\kappa(u,p)\nabla u + s(u,p)(u+p)\nabla p\big] = 0 \ \text{ in } \Omega \tag{45}$$

$$\nabla \cdot (s(u,p)\nabla p) = 0 \ \text{ in } \Omega \tag{46}$$

$$p = 0 \ \text{ on } \Gamma_0, \ p = q_1 \ \text{ on } \Gamma_1, \ q_1 \in \mathbf{R}^1, \ q_1 > 0 \tag{47}$$

$$u = 0 \ \text{ on } \Gamma. \tag{48}$$

For this problem when the viscosity and the thermal conductivity depend only on the temperature we refer to [12,13]. According to Lemma 2.1 the two-point problem corresponding to (45)-(48) is

$$\kappa(p,U)\frac{dU}{dp}+s(p,U)(U+p)=\lambda s(p,U) \tag{49}$$

$$U(0)=0,\ U(q_1)=0. \tag{50}$$

Since we have on physical ground $\kappa(p,U)\geq\kappa_0>0$, we can set $a(p,U)=s(p,U)/\kappa(p,U)$ and rewrite (49), (50) as follows

$$\frac{dU}{dp}+a(p,U)U=a(p,U)(\lambda-p) \tag{51}$$

$$U(0)=0,\ U(q_1)=0. \tag{52}$$

We have

Theorem 4.1. *If*

$$a(p,U)\in C^1([0,q_1]\times\mathbf{R}^1) \tag{53}$$

and

$$0<a_0\leq a(p,U)\leq a_1+a_2U \tag{54}$$

the two-point problem (51), (52) has one and only one solution.

Proof. We apply the Hadamard-Caccioppoli global inversion theorem [14]. Let

$$\mathcal{X}=\{U(p)\in C^1([0,q_1]),\ U(0)=0,\ U(q_1)=0\}\times\mathbf{R}^1,\ \mathcal{Y}=C^0([0,q_1])$$

and define the operator $\mathcal{F}:\mathcal{X}\to\mathcal{Y}$

$$\mathcal{F}(U,\lambda)=\frac{dU}{dp}+a(p,U)U-a(p,U)(\lambda-p).$$

The first differential of $\mathcal{F}$ computed in $(U, \lambda) \in \mathcal{X}$ and evaluated in $(H, \gamma) \in \mathcal{X}$ is given by

$$\mathcal{F}'(U, \lambda)[H, \gamma] = \frac{dH}{dp} + A(p)H - \gamma B(p)$$

where

$$A(p) = \frac{\partial a}{\partial U}(p, U(p))(U(p) - \lambda + p) + a(p, U(p)), \quad B(p) = a(p, U(p)).$$

We prove the invertibility of $\mathcal{F}'(U, \lambda)[H, \gamma]$ verifying the solvability for any $g(p) \in \mathcal{Y}$ of the problem

$$\frac{dH}{dp} + A(p)H - \gamma B(p) = g(p) \tag{55}$$

$$H(0) = 0 \tag{56}$$

$$H(q_1) = 0.$$

The solution of the Cauchy problem (55), (56) is given by

$$H(p; \gamma) = e^{-\int_0^p A(s)ds} \left[\int_0^p (\gamma B(\tau) + g(\tau)) e^{\int_0^\tau A(s)ds} d\tau \right].$$

By (54) we have $B(\tau) \neq 0$. Hence the equation $H(q_1; \gamma) = 0$ has one and only one solution. Thus (55), (56) is solvable and $\mathcal{F}'(U, \gamma)$ is invertible. We claim that $\mathcal{F}$ is a proper map, i.e. if $\mathcal{K}$ is a compact subset of $\mathcal{Y}$, $\mathcal{F}^{-1}(\mathcal{K})$ is a compact subset of $\mathcal{X}$. The functions $g(p)$ of $\mathcal{K}$ are uniformly bounded and equicontinuous, thus

$$|g(p)| \leq C_1(\mathcal{K}). \tag{57}$$

In $\mathcal{F}^{-1}(\mathcal{K})$ we have the solutions of the problem

$$\frac{dU}{dp} + a(p, U)U - a(p, U)(\lambda - p) = g(p) \in \mathcal{K} \tag{58}$$

$$U(0) = 0 \tag{59}$$

$$(60) \qquad U(q_1) = 0.$$

From (58) and (59) we obtain

$$(61) \qquad U(p) = \int_0^p \big[a(t, U(t))(\lambda - t) + g(t)\big] e^{-\int_t^p a(\tau, U(\tau)) d\tau} \, dt$$

and by (60)

$$(62) \qquad \lambda = \frac{\int_0^{q_1} [t a(t, U(t)) - g(t)] e^{-\int_t^{q_1} a(\tau, U(\tau)) d\tau} \, dt}{\int_0^{q_1} a(t, U(t)) e^{-\int_t^{q_1} a(\tau, U(\tau)) d\tau} \, dt}.$$

By (54) and (57) we obtain from (62)

$$(63) \qquad |\lambda| \leq q_1 + \frac{C_1}{a_0} = C_2.$$

We claim that the functions of $\mathcal{F}^{-1}(\mathcal{K})$ are uniformly bounded. For, from (61) and (63) we have

$$|U(p)| \leq \int_0^p \big[\big(a_1 + a_2|U(t)|\big)(C_2 + q_1) + C_1\big] e^{-\int_t^p a_0 d\tau} \leq \alpha + \beta \int_0^p |U(t)| dt,$$

where $\alpha = a_1(C_2+q_1)+C_1$, $\beta = a_2(C_2+q_1)+C_1$. By the Gronwall inequality we obtain

$$|U(p)| \leq \alpha e^{\beta q_1} = C_3,$$

where the constant C_3 depends only on the data and on $\mathcal{K}$. We prove that the functions of $\mathcal{F}^{-1}(\mathcal{K})$ are also equicontinuous. From (51), (54) and (63) we have

$$\Big|\frac{dU}{dp}\Big| \leq \big(a_1 + a_2|U|\big)\big(|\lambda| + q_1 + |U|\big) \leq \big(a_1 + a_2 C_3\big)\big(C_2 + q_1 + C_3\big).$$

This proves that $\mathcal{F}^{-1}(\mathcal{K})$ is a compact subset of $\mathcal{X}$. The Hadamard-Caccioppoli theorem applies and we conclude that problem (51), (52) has one and only one solution. □

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5. A special case reducible to the Riccati equation.

We consider here a special case of problem (45)-(48) for which the corresponding two-point problem has only and only one solution even if the assumption (53) of Theorem 4.1 is not satisfied. Let s and κ depend only on u and satisfy $s(u) = 2\sqrt{u}\kappa(u)$. Thus $a(U) = 2\sqrt{U}$. The problem (51), (52) becomes

$$\frac{dU}{dp} + 2\sqrt{U}U = 2\sqrt{U}(\lambda - p)$$

$$U(0) = 0, \; U(q_1) = 0.$$

If we define

$$U = \mathcal{U}^2$$

we have in terms of $\mathcal{U}$

$$\frac{d\mathcal{U}}{dp} + \mathcal{U}^2 + p = \lambda \tag{64}$$

$$\mathcal{U}(0) = 0, \; \mathcal{U}(q_1) = 0. \tag{65}$$

(64) is a Riccati equation [15]. This permits to solve the Cauchy problem (64), (65) in terms of the Airy functions $A(x)$, $B(x)$, see [16] . We obtain

$$\mathcal{U}(p) = \frac{A'(\lambda - p)B'(\lambda) - A'(\lambda)B'(\lambda - p)}{A'(\lambda)B(\lambda - p) - A(\lambda - p)B'(\lambda)}. \tag{66}$$

To discuss (66) we may limit to consider the case $\lambda > 0$. For, if $\lambda \leq 0$ and $\mathcal{U}'(p^*) = 0$ (such a $p^* \in (0, q_1)$ certainly exists in view of (65)), we have $p^* \leq 0$, a contradiction. We are interested in the solutions λ of the equation

$$\frac{N(\lambda, q_1)}{D(\lambda, q_1)} = 0$$

where

$$N(\lambda, q) = A'(\lambda - q)B'(\lambda) - A'(\lambda)B'(\lambda - q)$$

$$D(\lambda, q) = A'(\lambda)B(\lambda - q) - A(\lambda - q)B'(\lambda).$$

We study separately the solutions of the two systems

$$N(\lambda, q) = 0,\ \ q = q_1$$

$$D(\lambda, q) = 0,\ \ q = q_1$$

for $\lambda \geq 0$ and $q \geq 0$. Let

$$H = \{(\lambda, q) \in \mathbf{R}^2;\ N(\lambda, q) = 0,\ \lambda \geq 0,\ q \geq 0\}.$$

Under the transformation $x = \lambda - q$ the set H becomes

$$\mathcal{H} = \{(\lambda, x) \in \mathbf{R}^2;\ \mathcal{N}(\lambda, x) = 0,\ \lambda \geq 0,\ \lambda - x \geq 0\},$$

where $\mathcal{N}(\lambda, x) = A'(x)B'(\lambda) - A'(\lambda)B'(x)$ and similarly $\mathcal{D}(\lambda, x) = A'(\lambda)B(x) - A(x)B'(\lambda)$. We have

$$\frac{\partial^2 \mathcal{N}}{\partial x^2}(0,0) = -\frac{\partial^2 \mathcal{N}}{\partial \lambda^2}(0,0) = 0.31..,\ \frac{\partial^2 \mathcal{N}}{\partial x \partial \lambda}(0,0) = 0.$$

In the plane λ, x the point $(0,0)$ is a double point of the curve $\mathcal{N}(\lambda, x) = 0$ and the equation of the tangents in $(0,0)$ is $(x - \lambda)(x + \lambda) = 0$, $x = \lambda$ is part of the curve $\mathcal{N}(\lambda, x) = 0$ whereas $x = -\lambda$ is the tangent to the second branch emanating from $(0,0)$. We denote a_k, a'_k, b_k, b'_k the denumerable sequence of zeros of the functions $A(x)$, $A'(x)$, $B(x)$ and $B'(x)$ respectively counted from right to left. Moreover, n_k and d_k are the sequence of zeros of the functions $\mathcal{N}(0, x)$ and $\mathcal{D}(0, x)$ again counted from right to left. They are interlaced according to the following scheme for $k = 0, 1, 2, ...$

(67)
$$n_{k+1} < a_{k+1} < b'_{k+1} < d_{k+1} < b_{k+1} < a'_{k+1} < n_k < a_k < b'_k < d_k < b_k < a'_k < 0.$$

The equation $\mathcal{N}(\lambda, x) = 0$ can be rewritten

$$\frac{A'(x)}{B'(x)} = \frac{A'(\lambda)}{B'(\lambda)}$$

when $\lambda \geq 0$ and $x \neq b'_k$, $k = 0, 1$. Define

$$f(x) = \frac{A'(x)}{B'(x)}.$$

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The function $f(x)$ has, for $x < 0$, a "tangent-like" behaviour. Recalling that the Airy's functions are two linearly independent solutions of the equation $\frac{d^2y}{dx^2} = xy$ we have

$$f'(x) = \frac{x[A(x)B'(x) - B(x)A'(x)]}{(B'(x))^2}.$$

By the formula of Liouville we have

$$A(x)B'(x) - B(x)A'(x) = A(0)B'(0) - B(0)A'(0) > 0,$$

thus $f'(x) > 0$ if $x > 0$ and

$$f'(x) < 0 \text{ if } b_0' < x < 0, \text{ and } b_{k+1}' < x < b_n', \text{ k=1, 2...}$$

Moreover,

$$\lim_{x\to\infty} f(x) = 0$$

and, for $k = 0, 1, 2...$

$$\lim_{x\to b_0'+} f(x) = \infty, \lim_{x\to b_0'-} f(x) = -\infty, \lim_{x\to b_k'+} f(x) = \infty, \lim_{x\to b_k'-} f(x) = -\infty.$$

We are interested in the branches of solutions of $f(x) = f(\lambda)$ when $\lambda \geq 0$ and $x < 0$. With reference to Figure 1 we see that

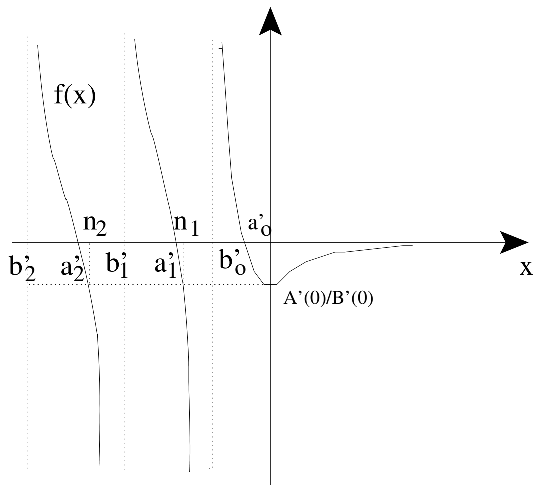

Figure 1.

(68) $$f([0, \infty)) = \Big[\frac{A'(0)}{B'(0)}, 0\Big)$$

and

$$f^{-1}\Big(\Big[\frac{A'(0)}{B'(0)},0\Big)\Big)=(a_0',0)\cup\cup_{k=1}^{\infty}(a_k',n_k]. \tag{69}$$

Hence the set

$$\mathcal{H}=\{(\lambda,x)\in\mathbf{R}^2;\ \mathcal{N}(\lambda,x)=0,\ x\geq 0,\ \lambda-x\geq 0\},$$

in addition to the half line $x=\lambda$, $\lambda\geq 0$, is composed of the graphs of infinite strictly decreasing functions $x=n_k(\lambda)$, $\lambda\geq 0$, $k=0,1,2...$ More precisely the part of $\mathcal{H}$ contained in the strip $[0,\infty)\times[a_0',0]$ is the graph of a function $x=n_0(\lambda)$ such that

$$n_0(0)=0,\ n_0'(\lambda)<0 \text{ for } \lambda>0,\ \lim_{\lambda\to\infty}n_0(\lambda)=a_0',$$

whereas the part of $\mathcal{H}$ contained in the generic strip $[0,\infty)\times[a_k',a_{k-1}]$, $k=1,2,..$ is the graph of a function $x=n_k(\lambda)$ such that

$$n_k(0)=n_k,\ n_k'(\lambda)<0 \text{ for } \lambda>0,\ \lim_{\lambda\to\infty}n_k(\lambda)=a_k'$$

(see Figure 2).

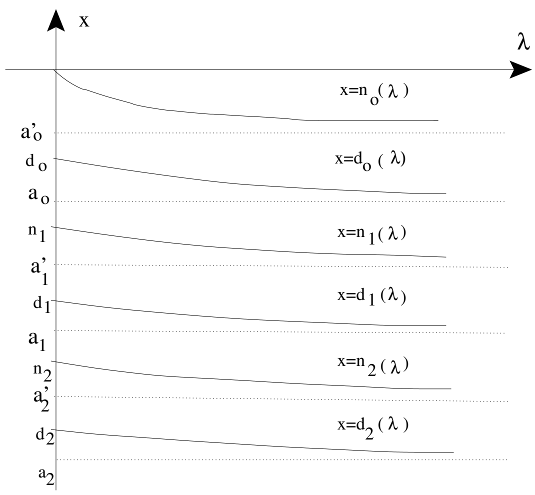

Figure 2.

We turn now to the study of $\mathcal{D}(\lambda,x)=0$. This is equivalent, if $\lambda\geq 0$ and $x\neq b_k$, to

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$$\frac{A'(\lambda)}{B'(\lambda)} = \frac{A(x)}{B(x)}.$$

Let

$$h(x) = \frac{A(x)}{B(x)}.$$

We have

$$h'(x) = \frac{A'(x)B(x) - B'(x)A(x)}{(B(x))^2}.$$

By the formula of Liouville $A'(x)B(x) - B'(x)A(x) = A'(0)B(0) - B'(0)A(0) < 0$. Thus, for $x \neq b_k$

$$h'(x) < 0. \tag{70}$$

The graph of $h(x)$ is plotted in Figure 3.

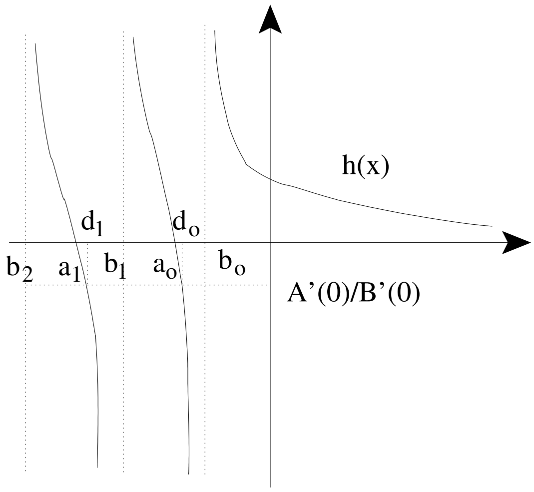

Figure 3.

Recalling (68) and that $\mathcal{D}(0, x) = 0$ means $A'(x)/B'(x) = A'(0)/B'(0)$ we have

$$h^{-1}\Big(\Big[\frac{A'(0)}{B'(0)}, 0\Big)\Big) = \cup_{k=0}^{\infty}(a_k, d_k). \tag{71}$$

By (70) each term of (71) is represented in the λ, x plane by the graph of a function $x = d_k(\lambda)$ such that

$$d_k(0) = d_k,\ d_k'(\lambda) < 0,\ \lim_{\lambda\to\infty} d_k(\lambda) = a_k,\ k = 0, 1, 2..$$

(see again the figure 2). Returning in the $\lambda,\ q$ plane we obtain, for the first branches of $N(\lambda, q) = 0$ and $D(\lambda, q) = 0$, the diagram of Figure 4.

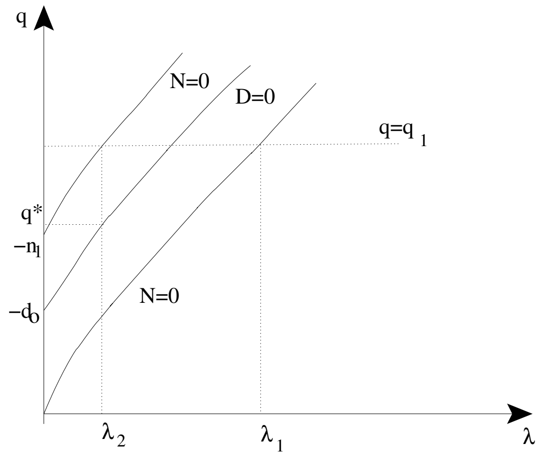

Figure 4.

If we fix $q_1 \in (0, \infty)$ the system $N(\lambda, q) = 0$, $q = q_1$ has a finite number of solutions, number which grows as q_1 grows to infinity. However, only the largest (λ_1 in the Figure 4) is acceptable since for all the others (like λ_2 in the Figure 4) the denominator $D(\lambda, q)$ of $U(\lambda, q)$ vanishes for a value $q^* \in (0, q_1)$ and is therefore to be rejected. Therefore the problem (64) and (65) has always one and only one solution.

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