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Research Paper

The homogeneity of two sets of European economies

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Dedicated to Professor Francesco Mainardi
on the occasion of his retirement

Communicated by Gianni Pagnini and Enrico Scalas

Abstract

It is generally accepted that an economy may be represented by a set of economic parameters. The normalised versions of these parameters are here assumed as coordinates in a Cartesian space. The homogeneity of a set of economies is then defined as the inverse of the sum of the distances of each economy from the others of the set (M. Caputo, *Meccanica*, 2014). The approach is applied to 2 sets of economies of the EU countries: 5 Western supposedly with evolved economies and 5 Eastern supposedly aiming to a better economic condition and using sets of 12 and 14 parameters respectively. It is found that, in the period 2000-2010, the homogeneity of both sets oscillates in time and that it does not seem that the 2 sets evolve to a common homogeneity. This is confirmed by the same type of analysis applied to the set of the 5 Western EU economies using 29 economic parameters of the year 2012 (M. Caputo, *Meccanica*, 2014) finding that the homogeneity of this set keeps oscillating.

Keywords: Dynamic, economies, memory, convergence, attractors.

AMS subject classification: 91B55, 26A33.

1. Introduction.

The economies of the EU are non homogeneous because of different cultures, labour markets, business intelligence, fiscal policy, wage gaps, standard living, governance size, administrative structures and traditions it is then natural to assume that the evolution of these economies occurs with a different pace [7].

The different paces of their bureaucracies and administrative structures are generally considered an internal friction which causes delays in the important processes of decision making, of planning, of loaning, and of acting.

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This phenomenon has already been considered in the economic literature and also taken into account by introducing a mathematical memory formalisms in the governing equation used in macroeconomy [5,8].

The discussion of the above mentioned phenomenon of the memory concerning the evolution of economies needs some premises which will precede the presentation of the method used for the estimate of the homogeneities of economies which, in turn, will be followed by the application of the method to 2 sets of EU economies and by the discussion of their homogeneities.

2. Premise on mathematical memory formalisms and their use in science.

The importance of memory functions is widely recognized in science. It is an important tool in the description of many phenomena in the fields of mathematical physics, engineering, informatics and biology.

Concerning the economy Galbraith, mostly referring to economic crises, stated: "the number of people restrained by memory is bound to decrease in time" [12], Demaria added: "a dynamic economy without memory is unthinkable" [9]. The same could be safely repeated for the evolution of populations. If this is reality, since memory phenomena are irreversible, so would be economic phenomena and history and the recovery after an economic crisis is actually an adjustment into a new path.

The recent economic crises in the EU, also in relation to the problems of the EU such as the different attitudes for stronger constraint on their monetary and fiscal policies, have renovated the interest on the problems of the evolutions of the countries of the Union [11] which, especially concerning the homogeneity, has been very recently addressed with econometric methods, and including memory formalisms, in the notes of Caputo [6,7].

There could be more than one way to represent the memory in mathematical form in the various fields of science, the fractional order derivative used in this note is defined as follows

$$D^{\upsilon}(t) = \frac{1}{\Gamma(1-\upsilon)} \int_0^t \frac{f'(u)}{(t-u)^{\upsilon}} du \qquad \upsilon \in \,]0,1[\tag{1}$$

where $\upsilon \in [0,1[$ is the order of differentiation and $\Gamma(\cdot)$ is the gamma function. The mathematical memory formalism defined in equation (1) fractional derivative, is discussed in many treatises; the most recent and readily available are e.g. [10,14–19].

The derivative of fractional order of $f(t)$ is constructed by taking a weighted mean of the first order derivative $[df(t)/dt]z$ in the time interval $[0,t]$, so as to induce a sort of feedback system. That is the values of

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$[df(t)/dt]z$ at time z far apart from t are given smaller weight than those at times z closer to t. Given that the weights are increasingly smaller with increasing time separation from time t, the effect of the past fades away as time goes by.

Notice that the weights multiplying the first order derivative of $f(t)$ inside the integral in equation (1) can be chosen in many ways. The definition adopted in equation (1) is algebraically simple, and allows easy analytic use.

Other mathematical methods of analysis of the evolution of economies, also in the non linear field, may be retrieved from the vast literature concerning this topic, among others the book of Gandolfo [13]. To my knowledge however I cautiously say that, in most cases with the linear approach, the different studies used the tool of the mathematical memory formalism called fractional derivative.

3. Premise on the evolution of economies.

The study of the evolution of economies is of interest particularly concerning the homogeneity of economies of the EU. A good introduction to the problems of economic growth, development and dynamics is in the last chapters the book of Samuelson [20] who quotes Smith, Malthus and Schumpeter. More recent; interesting works are those of Barro [2,3] and Barro and Sala-i-Martin [4], Benson and Johnson [5] who however concentrated their attention on growth.

Concerning the evolution of economies we may tentatively assume that an economic entity depends linearly on the difference between its states of evolution, measured in a given consistent method, and those of the other coexisting economic entities.

In the note of Caputo [6] was assumed that the rate of evolution of a set of economies be governed by a mechanisms of memory represented by fractional derivatives of order υ_i writing

$$D^{\upsilon_i} y_i(t) = \sum_{1}^{m} a_{ij}(y_j(t) - y_i(t)) \tag{2}$$

where is assumed that all υ_i are equal with $\upsilon_i \in]0, 1[$. In the note of Caputo [6], it was simply assumed that the y_i where the GDP of the i-economy and shown that the asymptotic values of all economies are equal and in the case of two economies with different order of fractional derivatives, converge to the asymptotic values of the economy with the smallest order of derivation.

In the note of Caputo [7] the result was generalized to the case when the order of fractional differentiation vi of the economies are different reaching the sane conclusions as in the case when the economies are. It is the economic entity with longer memory to dominate the pathway of all economic entities to the common asymptotic value, this economy acts as an attractor on the others. We note that formalism of the system (2) mimics Fourier equation and the law of attraction of elastic type.

This property of a set of different economies seems acceptable from the intuitive point of view: the economy less sensitive to the presence of the other economies could be the surviving one, the low sensitivity being represented by the long memory of its past. This could imply that the model requires a long period of stability and, in a short period, could be used only with great caution. Moreover, in reality, the economists have the opportunity to adjust the course of their economy acting on the appropriate structures in order to avoid possible extreme event, which could seriously disturb the stability required by the model.

For the purpose of investigating the evolution and the homogeneity of the EU economies and also to test the validity of the theoretical model of evolution we will use the method of their Cartesian distances.

4. Premise on the homogeneity of the economies.

The study of the evolution of the economies has been of great interest since long time. Presently is of particular interest the homogeneity of the countries of the EU which has been addressed with econometric methods also with the perspective of estimating the different attitudes and progress of the EU countries for an evolution towards a political union.

As we stated, the theoretical model would require a long period of stability in order to be of help studying the evolution of economies; a direct econometric approach would be required to complete its study and measure directly the evolution or the homogeneity of a set of economies or of sets of economies. However in spite of the short time interval of data available on the economies of EMU, in the present note, we tentatively try to study the homogeneity of sets of economies.

To this purpose we will use the measures of the distances between economies

obtained using a geometric method which allows to compare the single distances between one economy and the others of the set, their scatter, the sum of the distances of one economy with the others of the set and their scatter. We will use the period between the years 2000-2010 which is the longest available with the EMU data.

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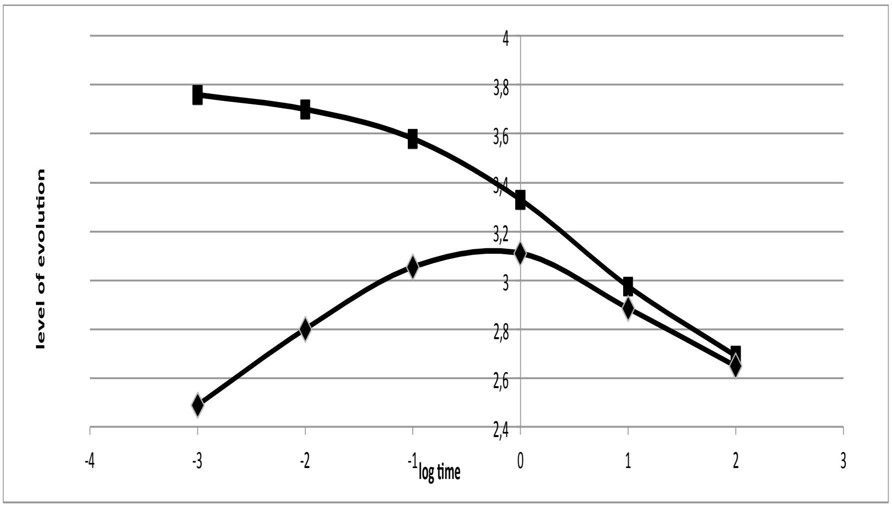

Figure 1. Evolution of two economic entities according to the memory model in [6]. Note that the initial value of one of the economy (diamonds), that with the fractional derivative of smaller order, is the attractor of the other economy (squares); both economies asymptotically converge to that initial value.

5. The homogeneity of the economies.

The econometric study of the homogeneity is here first addressed to 2 the sets of m economies with n parameters of the EU. As measure of the inhomogeneity between economies is assumed the algebraic distance in the Cartesian space of the parameters defining their charactestics.

The distances are obtained normalising each parameter p_j to the maximum value of its norm, acquiring a new set of normalised parameters q_j and considering the set $x_{ik,j}$ of the couples of difference of the normalised parameters q_j ; p_j is then substituted with

$$q_j = \frac{p_j}{|p_{jm}|} \tag{3}$$

where $q_j \leq 1$ defines a new Cartesian space.

We first assume the case when all parameters p_j have positive values and consider the differences

$$x_{ik,j} = q_{ij} - q_{kj} \tag{4}$$

with $|x_{ik,j}| \leq 1$, which are the components of the distance between the economy i dentified by k from that identified by i relative to the parameter j in the Cartesian space of the parameters q_j.

From the definition (3) follows that

$$\sum_{j=1}^{n}(\frac{p_{ij}-p_{kj}}{p_{j\,\max}})^2 < n \tag{5}$$

or

$$D_{ik} = \left[\sum_{j=1}^{n}(q_{ij}-q_{kj})^2\right]^{0.5} n^{-0.5} < 1 \tag{6}$$

where D_{ik} is the distance of the economies i and k. in the Cartestian space defined by the parameters q_j.

The normalizing factor of D_{ij} obtained considering first the case when all parameters assume non negative values and that m is even: consider now that if the values of the parameters of a given subset of $u < m$ of the m economies of the set are unity and all the others are zero, then the sum of all the $m(m-1)/2$ distances is $n^{0,5}u(m-u)$ whose maximum is obtained when $u = m/2$ which gives the distance $\frac{m^2 n^{0,5}}{4}$. If one, or more than one, of the zero value parameters were to assume a positive value the sum of the distances would decrease. The same applies also to the case when the values 1 are smaller than 1. The case when m is odd is obtained with the same procedure.

It is seen that when all parameters assume non negative values the sum of the distances Dik has the maximum

$$\begin{aligned} & n^{0,5}\frac{m^2}{4}, && \text{when } m \text{ is even} \\ & n^{0,5}\frac{(m^2-1)}{4}, && \text{when } m \text{ is odd} \end{aligned} \tag{7}$$

Finally, taking into account the possible presence r parameters which may assume negative values and that the corresponding values of are subject to the limit, formulae (7) are modified to

$$\begin{aligned} U &= \frac{D_{ik}}{(n+3r)^{0,5}m^2/4} && \text{when } m \text{ is even} \\ U &= \frac{D_{ik}}{(n+3r)^{0,5}(m^2-1)/4} && \text{when } m \text{ is odd} \end{aligned} \tag{8}$$

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where $U \in [0,1]$ could tentatively be considered as the tentative abstract measure of the inhomogeneity of the set of economies. The homogeneity of the set is then inversely proportional to the value of U.

Obviously the distances obtained are only abstract tools and so far we may compare the different economies with the understanding that larger values of U imply relevant differences in the economies.

The meaning of U would specially be in the comparison of its values obtained at subsequent times; these values may indicates if the members of the set of economies is becoming less or more homogeneous, that is, converge to a unique state when all the parameters are theoretically equal.

6. The time variation of the homogeneity of two sets of EU economies.

In order to explore the evolution of the homogeneity of the EU economies we now consider two sets of economies:

A) France, Germany, Italy, Spain the UK

B) Bulgaria, Czech Republic, Hungary, Poland, Romania,

which, for historic reasons, have different economies and, obviously, should have different economic evolutions and, perhaps, different homogeneity. The study of the evolution and homogeneity of the economies in each set is carried out examining the variations of the state of the 2 sets in time. For this purpose was selected a set of similar parameters for the years 2010, 2005 and 2010 taken from EUROSTAT and listed in the appendix B to be normalized and used as Cartesian orthogonal coordinates.

The results concerning the 2 sets are shown in the following tables 1 and 2 where m is the number of economies of the set, n is the number of parameters used, r is the number of parameters which may assume negative values and U $[0,1]$ is the normalized measure of inhomogeneity. The "sum" is the sum of the distances of each economy from the others of the set, the "SUM" is the sum distances between all the economies of the set.

The spread of the distance of each couple of economies does not seem large but it varies in time which could imply some sort of instability of the economies which is confirmed by the time variation of the level of inhomogeneity U.

We note in the Fig. 2 that the distance of the Italian economy from the others is increasing from 2000 to 2010 as if the changes in Europe had a negative effect on it, while the distances of all the others oscillate first increasing then decreasing with different paces.

Table 1. Set A.

2000					
	Fr	gr	it	sp	Uk
Fr		0,76	0,64	1,14	2,49
Gr	0,76		1,05	1,53	2,01
It	0,64	1,05		0,97	1,69
Sp	1,14	1,53	0,97		2,07
Uk	**2,05**	**2,01**	**1,69**	**2,07**	
Sum	4,59	5,35	4,35	5,71	7,78
rel st dev	0,14	0,1	0,1	0,085	0,042
0,378	= U	m=5	n=12	r=1	
SUM	13,92141	rel st dev	0,977		
2005					
	Fr	gr	it	Sp	Uk
Fr		1,16	0,61	1,49	2,33
Gr	1,16		1,55	2,08	2,67
It	0,61	1,55		1,8	2,31
Sp	1,49	2,08	1,8		2,45
Uk	**2,33**	**2,67**	**2,31**	**2,45**	
Sum	5,59	7,46	6,27	7,82	9,76
rel st dev	0,13	0,088	0,11	0,052	0,017
0,476	= U	m=5	n=12	r=1	
SUM	18,47	rel st dev	0,0352		
2010					
	Fr	gr	It	Sp	Uk
Fr		1,04	1,5	1,03	0,87
Gr	1,04		2,25	1,86	1,32
It	1,5	2,25		1,84	1,97
Sp	1,03	1,86	1,84		1,28
Uk	0,87	1,32	1,97	1,28	
Sum	4,44	6,47	7,56	6,01	5,44
rel st dev	0,073927	0,077279	0,039541	0,064463	0,083581
0,352	= U		m = 5	n =12	r = 2
SUM	14,97	rel st dev	0,0776		

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Table 2. Set B.

2000					
	rom	ung	Pol	Bul	Cze
Rom		1,9	1,59	1,58	1,55
Ung	1,9		1,93	1,31	1,53
Pol	1,59	1,93		1,66	1,14
Bulg	1,58	1,31	1,66		2,14
Cze	1,55	1,53	1,14	2,14	
sum dist	6,62	6,67	6,32	6,69	6,36
dev st rel	0,024806	0,04495	0,051887	0,051685	0,064841
SUM	14,61082	dev st rel	0,0121		
R=1	U =	0,354	m = 5	n = 14	
2005					
	rom	ung	Pol	Bul	Cze
Rom		1,638	1,98	1,43	1,785
Ung	1,638		1,484	1,573	1177
Pol	1,98	1,484		2,201	1,383
Bulg	1,43	1,573	2,201		1,599
Cze	1,785	1,177	1,383	1,599	
sum dist	6,833	5,872	7,048	6,803	5,944
SUM	16,25			dev st rel	0,0338
r = 0	U =	0,434	m = 5	n = 14	
2010					
	rom	Ung	Pol	Bul	Cze
Rom		1,83	1,83	1,23	2
Ung	1,83		1,78	1,11	1,58
Pol	1,83	1,78		1,45	1,36
Bulg	1,23	1,11	1,45		1,54
Cze	2	1,58	1,36	1,54	6,48
sum dist	6,89	6,3	6,42	5,33	6,48
SUM	15,71			st dev rel	0,0364
r = 0	U =	0,420	m = 5	n = 14	

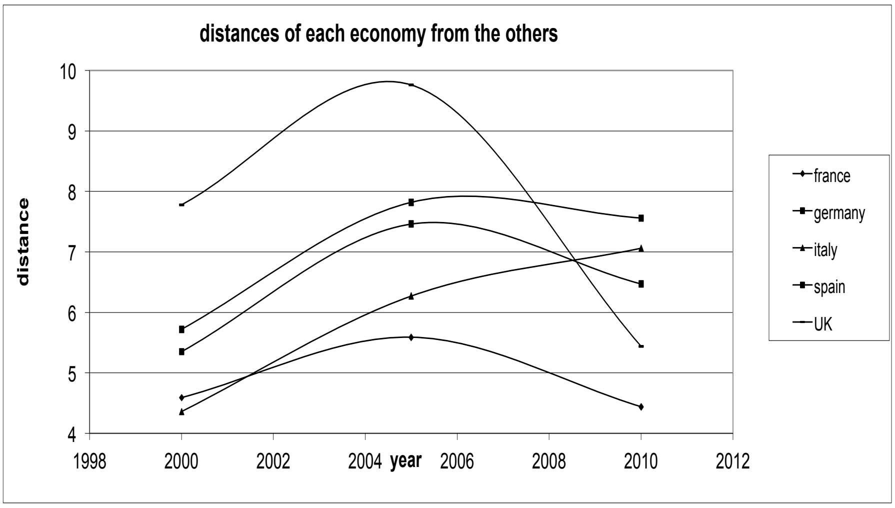

Figure 2. Distance of each economy of the set A from the others.

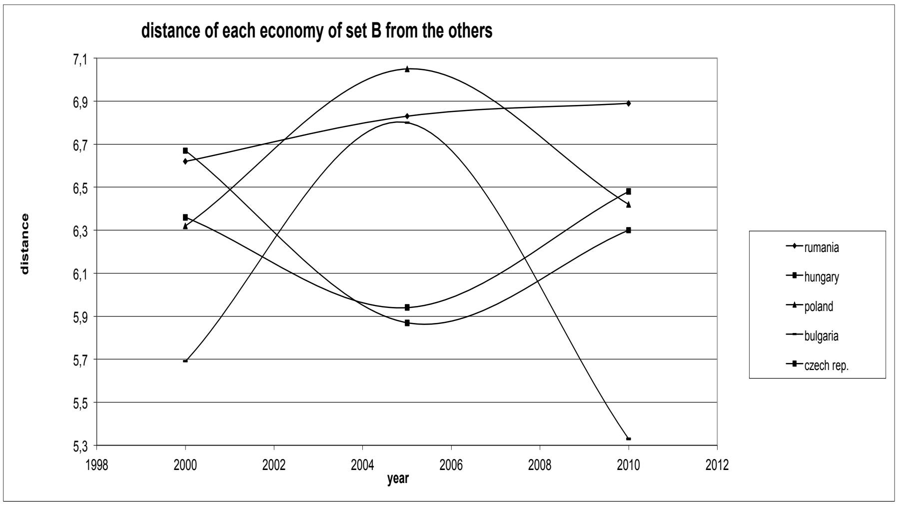

Figure 3. Distance of each economy of the set B from the others.

We note in the Fig. 3 that the distance of Romania from the others is very slowly increasing with time as if it were insensitive to the changes in Europe, while those of the others are oscillating first increasing then decreasing or vice versa with different paces as if the changes in Europe had opposite effect on their development.

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Note that the comments on the results shown in the tables 1 and 2 obviously refer to the sets of parameters used. The different number of parameters used in the 2 sets, 12 in set of economies A and 14 in the set B, is taken into account in the normalization needed to estimate the parameter U. We also assumed that the most of the important parameters are present in both sets. The results of the table 1 and 2 concerning the sets A and set B indicate that:

1) The SUM of distances between the economies of sets A and B are first increasing then decreasing as seen in the Fig. 2 and examining the corresponding values of U. This indicates that in the initial part of the period examined, which coincides with the introduction of the monetary union, the effect of this union was similar in the two sets.

2) Remembering that the increase of the parameter U implies an increase of inhomogeneity then both sets had an initial decrease of homogeneity, which however was, partly, later recovered in the set B and fully recovered in set A in the subsequent period 2005-2010.

3) The oscillations of the homogeneity of the two sets indicate that they are not yet on the path of reaching an equilibrium as would be suggested by the mathematical model of evolution presented, however the initial effects of the monetary union and the economic crisis occurring in the second part of the period examined are exogenous perturbations of the systems which deviated the economies from their natural evolutions. We may only note that the measures of the inhomogeneities of the 2 sets examined in the year 2010 ($U = 0.352$ for set A and $U = 0.420$ for set B) are not close since they are separated by 18% of their average. It would seem that, considering that the time interval monitored and the perturbations occurred in it, a check on the validity of the theoretical model of evolution considered in the first part of this note may not yet be possible.

4) The relative standard deviations in the two sets are about of the same order of magnitude of few percent which would confirm the presumed accuracy of the procedure used.

5) We note that there is only a vague similarity in the Figs. 4 and 1. Due to the short duration of the period monitored and the limited resolution we could not expect to see which one of the two sets could be considered the attractor.

6) Although more resolution could give better results is worth noting in the Figs.s 2 and 3 that the economies of Italy and Rumania are steadily increasing in time their distance from the others of their respective set, while the other economies of the sets oscillate.

7) The recovery in homogeneity in the period (from 2005 to 2010) is

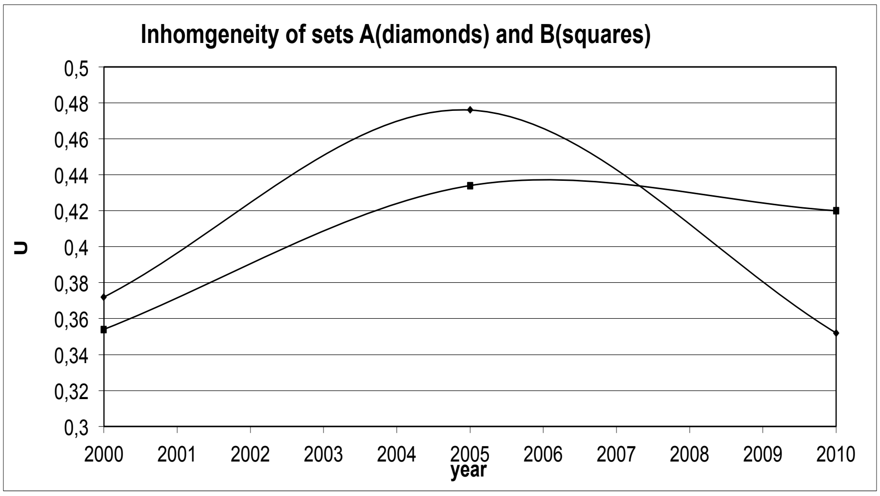

Figure 4. Inhomogeneity U of the two sets of economies analyzed. Squares are for set B and diamonds are for set A. The 2 sets seem to oscillate and seem to not be on a path to homogeneity.

encouraging, particularly for the set A, but the desirable convergence of the two sets is lacking.

7. Comparison with the results obtained using different parameters.

For the sake of checking the stability of the results when changing the parameters in the estimate of the distances we now tentatively compare the results of the table 1 for the set A with those shown in the table 3 obtained obtained for the same set using 29 parameters listed in the appendix A.

Concerning the use of different parameters we consider the set A for the year 2010 shown in table 1 made using 12 parameters (listed in the appendix A) giving the SUM = 14.97 and $U = 0.359$ compared with the analysis for the year 2012 shown in table 3, using 29 parameters (listed in the appendix B), giving SUM = 21.85 and $U = 0.406$.

But the SUM = 21.85 results from 29 parameters and, in order to compare it with the corresponding SUM = 14.97 from table 1 for the year 2010, we should normalize it relative to the number used for its evaluation, finding a sum of SUM = 16.59, not very far from the other value 14.97 obtained using 12 parameters. The values of U however are very close thus confirming the measure the homogeneity.

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Table 3. Distances between the economies of the set. The total indicates the sum of the distances of the economy relative to the others. RSD is the relative standard deviation. The sum of all the 10 distances is 21.85 and the corresponding $U = 0.406$.

	Spain	Italy	France	Germany	UK
Spain		1,88	2,5	2,93	2,25
Italy	1,88		2,19	2,21	2,11
France	2,5	2,19		1,72	2,15
Germany	2,93	2,21	1,72		1,91
UK	2,25	2,11	2,15	1,91	
TOTAL	9,56	8,39	8,56	8,77	8,42
RSD	5%	2%	4%	6%	1,70%

Finally we like to note that the density distribution of the $x_{ik,j}$ is bell shaped and, to a good approximation, has a Gaussian distribution as shown in the Fig. 3.

We note that if we change the order of the terms when computing $x_{ik,j}$ then each value of the set changes sign, the density distribution becomes symmetric relative to the preceding one thus conserving its presumed Gaussian density distribution and the standard deviation are the same.

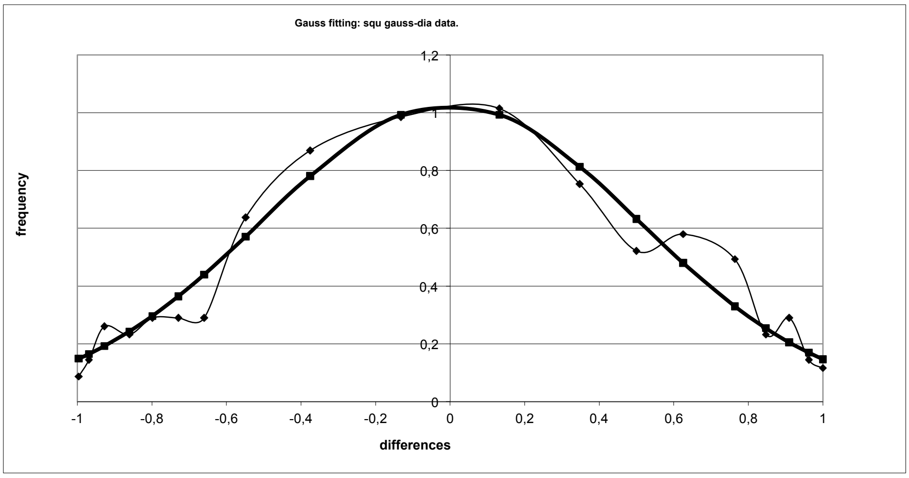

Figure 5. Density distribution and fitting to a Gaussian curve (squares) of the set formed with all 290 values of $x_{ik,j}$ (diamonds) obtained from the sets of 29 positive parameters of the economies of France, Italy, Germany, Spain and UK, defined by formulae (3) and (4).

The fit of the density distribution of the $x_{ik,j}$ to a Gaussian curve; gives a R2 factor value 0.83, somewhat short of being satisfactory, is shown in the Fig. 3. However is remarkable that so much information be stored in a set of economies parameters with Gaussian distribution.

8. Conclusions.

A possible interpretation of the mathematical memory based model of this note is that it may indirectly represent the adminitsrative and bureaucratic procedure, causing energy dissipation and internal friction as delaying factors in the evolution of the economy. This eventually takes into account, in an somewhat abstract form, also the second principle of thermodynamics, which takes all systems of nature to the same energy level by energy dispersal as suggested by Annila and Salthe [1]). However more comprehensive interpretations in the very complex field of economy is desirable.

A limit to the method used in this note for the estimate the homogeneity of economies is represented by the fact that it would not say about the internal economic equilibrium of the single economies of the set nor about the effects of exogenous perturbations which however are not the scope of the present note. However, using few, appropriately selected, sets of parameters, the method could possibly identify the parameters responsible for the inhomogeneity of the economies [7].

The knowledge of the evolution could possibly be of practical interest since those responsible of the economies would have the possibilities to correct the course of the evolution by changing the appropriate structures.

We must note that the measure of the distances used depends on the parameters selected but also on the number of economies in the sets and therefore would rigorously be comparable only the results of the analysis of sets with the same number of economies and parameters or, in a comparison of results obtained with different number of parameters and/or number of economies, one must apply the necessary normalizations using formulae (8).

We also note that an appropriate level of inhomogeneity could possibly be useful for a healthy evolution of the systems of economies; an interesting and challenging problem would then be to investigate which could be an appropriate physiological level of inhomogeneity.

Finally we cannot restrain ourselves to comment that the theories aiming to equilibrium are very useful in spite of the fact that, in practice: exist only the setting of the economies to the modifying conditions due to all kinds of crises of the world which, as we noted, is a limit also to the theoretical model presented in the first part of this study.

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Appendix A. Parameters used for Tables 3; values from C.I.A. factbook for 2012.

1) GDP per capita, 2) Agriculture product/GDP, 3) Industrial product/GDP, 4) Services/GDP, 5) Labour force, 6) Labour force industry %, 7) Labour force agriculture %, 8) Labour force services %, 9) People below poverty %, 10) Investment/GDP, 11) Tax/GDP, 12) Debt/GDP, 13) Inflation rate, 14) Lending prime of commercial banks, 15) Unemployment rate, 16) Exports, 17) Imports, 18) Reserves, 19) Debt foreign, 20) Electricity exports, 21) Electricity imports, 22) Electricity from fossil fuel, 23) Electricity from nuclear fuel, 24) Hydroelectric energy, 25) Electricity from renewable sources, 26) Gas consumption, 27) Natural gas exports, 28) Natural gas imports, 29) Electricity consumption.

Appendix B. Parameters used for tables 1 and 2; 2010 values from EUROSTAT.

The club A (France, Germany, Italy, Spain, UK) has the parameters:

1)GDP in $, 2) rate of growth, 3) population, 4) labour force/population, 5) inflation rate, 6) GDP per capita, 7) rate industrial growth, 8) public debt/GDP, 9) agriculture/GDP, 10) electricity consumption per capita, 11) oil consumption per capita, 12) Unemployment rate.

The club B (Poland, Czech Republic, Hungary, Romania, Bulgaria) has the parameters:

1)population, 2) GDP $, 3) GDP per capita, 4) labour force/population, 5) unemployment rate, 6) inflation rate, 7) rate industrial growth, 8) external debt /GDP, 9) electricity consumption per capita, 10) oil consumption per capita, 11) agriculture/GDP, 12) rate of growth, 13) investment/GDP, 14) public debt/GDP.

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