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Research Paper

Stochastic processes related to time-fractional diffusion-wave equation

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Dedicated to Professor Francesco Mainardi on the occasion of his retirement

Communicated by Gianni Pagnini and Enrico Scalas

Abstract

It is known that the solution to the Cauchy problem:

$$D_*^\beta u(x,t) = R^\alpha u(x,t)\,, \quad u(x,0) = \delta(x)\,, \quad \frac{\partial}{\partial x} u(x,t=0) \equiv 0\,, \quad -\infty < x < \infty\,, \quad t > 0\,,$$

is a probability density if $1 < \beta \le \alpha \le 2$, where D_*^β is the time fractional Caputo derivative of order β whereas R^α denotes the spatial Riesz fractional pseudo-differential operator. In the present paper it is considered the question if $u(x,t)$ can be interpreted in a natural way as the sojourn probability density (in point x, evolving in time t) of a randomly wandering particle starting in the origin $x = 0$ at instant $t = 0$. We show that this indeed can be done in the extreme case $\alpha = 2$, that is $R^\alpha = \frac{\partial^2}{\partial x^2}$. Moreover, if $\alpha = 2$ we can replace D_*^β by an operator of distributed orders with a non-negative (generalized) weight function $b(\beta)$: $\int_{(1,2]} b(\beta)\, D_*^\beta \ldots d\beta$. For this case $u(x,t)$ is a probability density.

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AMS subject classification: `26A33, 35L05, 45K05, 60E07.`

1. Introduction.

One dimensional time-fractional wave equation reads [1–4]

$$(1) \qquad D_*^\beta u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2}\,, \quad -\infty < x < \infty\,, \quad t > 0\,, \quad 1 < \beta \le 2\,,$$

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where D_*^β is the time-fractional derivative in the Caputo sense of order β ($m-1<\beta\le m$, $n\in I\!N$) that, for a sufficiently well-behaved function $f(t)$, is defined as

$$D_*^\beta f(t) = \begin{cases} \dfrac{1}{\Gamma(m-\beta)}\displaystyle\int_0^t \dfrac{f^{(m)}(\tau)\,d\tau}{(t-\tau)^{\beta+1-m}}\,, & m-1<\beta<m\,, \\ \dfrac{d^m}{dt^m} f(t)\,, & \beta = m\,, \end{cases} \tag{2}$$

where $\Gamma(\cdot)$ is the Gamma function. With the initial conditions $u(x,0)=\delta(x)$ and $\dfrac{\partial}{\partial t}u(x,t=0)=0$ the fundamental solution of (1) can be expressed in terms of the M-Wright/Mainardi function $M_\nu(z)$, $0<\nu<1$, [2–6]

$$u(x,t) = \frac{1}{2}\, t^{-\beta/2} M_{\beta/2}(|x| t^{-\beta/2})\,, \tag{3}$$

where

$$M_\nu(z) = \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{(-z)^{n-1}}{(n-1)!}\Gamma(\nu n)\sin(\pi\nu n)\,. \tag{4}$$

The classical fundamental solution of the D'Alembert wave equation is recovered in the limit $\beta=2$. The interest reader can find historical notes and reviews on the M-Wrigth/Mainardi function $M_\nu(z)$ in References [5–8].

It is known that the fundamental solution $u(x,t)$ given in (3) can be interpreted as a probability density function in x evolving in time t, and here noted by $g(x,t)=u(x,t)$. Moreover, that there exists a random walk process such that $g(x,t)$ is the sojourn density of a randomly wandering particle that moves monotonically rightwards with the probability $1/2$ and monotonically leftwards with the probability $1/2$.

Gorenflo, Luchko and Stojanović [9] have shown that the solution $u_{\mathrm{d}}(x,t)$ to the Cauchy problem

$$\int_{(1,2]} b(\beta) D_*^\beta u_{\mathrm{d}}(x,t)\,d\beta = \frac{\partial^2 u_{\mathrm{d}}(x,t)}{\partial x^2}\,, \quad -\infty<x<\infty\,, \quad t>0\,, \tag{5}$$

with the initial conditions $u_{\mathrm{d}}(x,0)=\delta(x)$ and $\dfrac{\partial}{\partial t}u_{\mathrm{d}}(x,t=0)=0$ is a probability density in the spatial variable x evolving in the time variable t where $b(\beta)$ is a non-negative measure ($b(\beta)\ge 0$) and a generalized function in the sense of Gel'fand and Shilov [10] such that $0<\displaystyle\int_{(1,2]} b(\beta)\,d\beta<\infty$.

A noteworthy special case is $b(\beta) = \delta(\beta - \beta_0)$, which reduces Equation (5) to the single order time-fractional wave equation (1).

By using Laplace and Fourier transforms defined respectively as:

$$(6) \qquad \widetilde{f}(s) = \int_0^\infty f(t)\exp(-st)\,dt\,, \quad \widehat{g}(\kappa) = \int_{-\infty}^\infty g(x)\exp(+i\kappa x)\,dx\,,$$

and setting $B(s) = \displaystyle\int_{(1,2]} b(s)s^\beta\,d\beta$ we obtain

$$(7) \qquad \widehat{\widetilde{u}}_{\rm d}(\kappa, s) = \frac{B(s)/s}{B(s)+\kappa^2}\,.$$

From the spatial Fourier inversion it follows

$$(8) \qquad \widetilde{u}_{\rm d}(x, s) = \frac{\sqrt{B(s)}}{2\,s}\exp(-|x|\sqrt{B(s)})\,.$$

By aid of the basic theory of complete monotone-, Stieltjes-, Bernstein- and complete Bernstein-functions (see [11]), in [9] is concluded that $u_{\rm d}(x,t)$ indeed is a probability density in x evolving in t, and hereinafter it is noted by $q(x,t) = u_{\rm d}(x,t)$.

Moreover, in [9] it is also raised the question for existence of a stochastic process so that $q(x,t)$ is the sojourn probability density for a wandering particle to be in point x at instant t.

During the International Symposium on Fractional PDEs: Theory, Numerics and Applications, Salve Regina University, Newport, RI, USA, June 3–5, 2013, the present author conveyed this question to the audience as a challenge and then had illuminating discussions on it with Mark M. Meerschaert (Michigan State University, USA) who supposed that $\exp(-\sqrt{B(s)})$ is a Laplace transform of an infinitely divisible distribution.

We will see in the following that $q(x,t) = u_{\rm d}(x,t)$ represents the spatial density of a stochastic process to be in x at time t.

2. Solution of an open problem concerning the distributed order fractional wave equation.

From the definition, we have $\sqrt{B(0)} = 0$ and in [9] it has been shown that $\sqrt{B(s)}$ is a complete Bernstein function. Hence, by Theorem 1 in chapter XIII.7 of [12] (see also [11]), function $\exp(-\sqrt{B(s)})$ is a Laplace transform of an infinitely divisible distribution (on the positive half-line), thus suited as a subordinator.

The inverse Laplace transform of $\exp(-x\sqrt{B(s)})$ is a density $r(t,x)$ (in $t \geq 0$, evolving in $x \geq 0$) of an increasing stochastic process $t = t(x)$ on the positive t-half-line starting in the origin $t = 0$ at instant $x = 0$. Note that here the roles of space and time are interchanged. Because of monotonicity of $t = t(x)$ we can determine the inverse (also increasing) process $x = x(t)$ and its density $q(x,t)$, a density in $x \geq 0$ evolving in time $t \geq 0$. These two processes are represented by the same graph if we visualize intervals of constancy and jumps by corresponding horizontal or vertical segments.

Now consider a fixed sample trajectory $t = t(x)$ and its likewise fixed inversion $x = x(t)$. For fixed $\bar{t}$ and fixed $\bar{x}$, because of monotonicity, we have the following relationships $x(\bar{t}) \geq \bar{x} \Leftrightarrow \bar{t} \geq t(\bar{x})$, and corresponding probabilities coincide. By the replacements $x(\bar{t}) \Rightarrow x'$, $\bar{x} \Rightarrow x$, $\bar{t} \Rightarrow t$, $t(\bar{x}) \Rightarrow t'$ we now arrive for the densities $r(t,x)$ and $q(x,t)$ at the relation

$$\int_0^t r(t',x)\,dt' = \int_x^\infty q(x',t)\,dx'\,, \tag{9}$$

hence

$$q(x,t) = -\frac{\partial}{\partial x}\int_0^t r(t',x)\,dt'\,. \tag{10}$$

Applying the Laplace transform gives

$$\widetilde{q}(x,s) = -\frac{\partial}{\partial x}\left\{\frac{1}{s}\,\widetilde{r}(s,x)\right\} = -\frac{1}{s}\frac{\partial}{\partial x}\exp(-x\sqrt{B(s)})\,, \tag{11}$$

and finally

$$\widetilde{q}(x,s) = \frac{\sqrt{B(s)}}{s}\exp(-x\sqrt{B(s)})\,, \quad \text{for} \quad x \geq 0\,. \tag{12}$$

3. An alternative approach.

Let us first consider the Cauchy problem for the single-order time-fractional wave equation

$$D_*^\beta u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2}\,, \quad -\infty < x < \infty\,, \quad t > 0\,, \quad 1 < \beta \leq 2\,, \tag{13}$$

with initial conditions $u(x,0) = \delta(x)$ and $\dfrac{\partial}{\partial t}u(x,t=0) = 0$. Setting $\nu = \dfrac{\beta}{2}$ we can split its Fourier–Laplace solution

$$\widehat{\widetilde{u}}(\kappa,s) = \frac{s^{\beta-1}}{s^\beta + \kappa^2} \tag{14}$$

into two terms:

$$\widehat{\widetilde{u}}(\kappa,s) = \frac{1}{2}\left(\frac{s^{\nu-1}}{s^\nu + i\kappa} + \frac{s^{\nu-1}}{s^\nu - i\kappa}\right). \tag{15}$$

Here the second term $\dfrac{s^{\nu-1}}{s^\nu - i\kappa}$ represents the density (in x, evolving in t) of a positive-oriented fractional drift process starting in the origin $x = 0$ at $t = 0$ with monotonically increasing sample paths. The corresponding Cauchy problem is

$$D_*^\nu r(x,t) = -\frac{\partial}{\partial x} q(x,t)\,, \quad q(x,0) = \delta(x)\,. \tag{16}$$

To treat it conveniently by Laplace–Laplace analysis we introduce (in addition to the Laplace transform with respect to t) also the Laplace transform with respect to x as

$$\breve{g}(\kappa) = \int_0^\infty g(x)\exp(-\kappa x)\,dx\,. \tag{17}$$

From $s^\nu \breve{\widetilde{q}}(\kappa,s) - s^{\nu-1} = -\kappa\,\breve{\widetilde{q}}(\kappa,s)$ we obtain

$$\breve{\widetilde{q}}(\kappa,s) = \frac{s^{\nu-1}}{s^\nu + \kappa}\,, \tag{18}$$

the Laplace–Laplace representation of the inverse ν-stable subordinator. Its relation to the positive-unilateral ν-stable density is visible in the partial inversion $\widetilde{q}(x,s) = s^{\nu-1}\exp(-xs^\nu)$. Its explicit form is given by the M-Wright/Mainardi function (see [6]):

$$q(x,t) = t^{-\nu} M_\nu(x\,t^{-\nu})\,. \tag{19}$$

A stochastic interpretation is also possible. The fundamental solution to the fractional wave equation of single order $\beta \in (1,2]$ describes a process where a particle starting in $x = 0$ decides with probability $1/2$ to wander in positive or negative direction and keeps this direction, thereby following the inverse $\dfrac{\beta}{2}$−stable subordinator (or its reflected process, respectively).

Analogously we now treat the case of distributed orders by splitting. We find

$$\widehat{\widetilde{u}}(\kappa,s) = \frac{B(s)/s}{B(s)/s + \kappa^2} = \frac{1}{2}\left(\frac{\sqrt{B(s)}/s}{\sqrt{B(s)} + i\kappa} + \frac{\sqrt{B(s)}/s}{\sqrt{B(s)} - i\kappa}\right). \tag{20}$$

Let us state the following analogy: with reference to formula (15) function $B(s)$ corresponds to s^β such that $\sqrt{B(s)}$ corresponds to $s^{\beta/2} = s^\nu$. Again we have a positive-oriented and a reflected negative-oriented process, each chosen with probability $1/2$ by a wandering particle, starting at the origin $x(0) = 0$. Denoting the positive-oriented process again by $q(x,t)$, we still can work with Laplace–Laplace transform in place of Fourier–Laplace transform and obtain in the transform domain

$$(21) \qquad \sqrt{B(s)}\,\breve{\widetilde{q}}(\kappa,s) + \kappa\,\breve{\widetilde{q}}(\kappa,s) = \frac{\sqrt{B(s)}}{s}\,, \quad \breve{\widetilde{q}}(\kappa,s) = \frac{\sqrt{B(s)}/s}{\sqrt{B(s)}+\kappa}\,.$$

By partial inversion we again get equation (12), namely

$$(22) \qquad \widetilde{q}(x,s) = \frac{\sqrt{B(s)}}{s}\exp(-x\sqrt{B(s)})\,, \quad \text{for} \quad x \ge 0\,.$$

Question:
Can the operator $\mathcal{A}$ whose Laplace symbol is $\sqrt{B(s)}$ be analyzed with the theory developed by Kochubei [13]?

Suggestion:
We conjecture that quite analogous results on interpretation as combination of two uni-directional diffusion processes also hold in the the more difficult case $1 < \beta \le \alpha < 2$. For this more difficult situation we suggest to investigate a subordination formula derived in [14] in which $u(x,t)$ is related to the inverse-stable subordinator of order β/α and the fundamental solution of the neutral fractional diffusion equation in which $\beta = \alpha$:

$$(23) \qquad \frac{1}{\pi}\frac{|x|^{\alpha-1}\,t^\alpha\sin(\pi\alpha/2)}{t^{2\alpha}+2\,|x|^\alpha\,t^\alpha\cos(\pi\alpha/2)+|x|^{2\alpha}}\,.$$

This function has been investigated in detail by Luchko in [15].

4. Conclusion.

This result allows us to interpret $u_{\mathrm{d}}(x,t)$ in equation (5) as sojourn density (at x, evolving in t) of a randomly moving particle, starting at the origin $x(0) = 0$, deciding with probability $1/2$ to move in positive or negative direction and then keeping to this decision. The motion in positive direction has density $q(x,t)$, that in negative direction has density $q(-x,t)$. Physically, when many particles are starting simultaneously, half of them move (in probabilistic sense) in positive direction, the other half in negative direction. Macroscopically we get two dispersing waves. This is the physical meaning of time-fractional wave. Simultaneously we have the character of wave and diffusion.

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