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Research Paper

Analysis and numerics of traveling waves for asymmetric fractional reaction-diffusion equations

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Dedicated to Professor Francesco Mainardi on the occasion of his retirement

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Abstract

We consider a scalar reaction-diffusion equation in one spatial dimension with bistable nonlinearity and a nonlocal space-fractional diffusion operator of Riesz-Feller type. We present our analytical results on the existence, uniqueness (up to translations) and stability of a traveling wave solution connecting two stable homogeneous steady states. Moreover, we review numerical methods for the case of reaction-diffusion equations with fractional Laplacian and discuss possible extensions to our reaction-diffusion equations with Riesz-Feller operators. In particular, we present a direct method using integral operator discretization in combination with projection boundary conditions to visualize our analytical results about traveling waves.

Keywords: Traveling wave, Nagumo equation, real Ginzburg-Landau equation, Allen-Cahn type equation, Riesz-Feller operator, nonlocal diffusion, fractional derivative, comparison principle, quadrature, projection boundary conditions.

AMS subject classification: 35A01, 35A09, 35B51, 35R09, 47G20.

1. Introduction.

A scalar reaction-diffusion equation is a partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(u) \quad \text{for } (x,t) \in \mathbb{R} \times (0,T]\,, \tag{1}$$

where the spatial derivative models diffusion and (a nonlinear) function f models reaction of some quantity $u = u(x,t)$ over time. The application and analysis of reaction-diffusion equations has a long history [1–3].

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In the following, we consider equation (1) with a *bistable* nonlinear function $f \in C^1(\mathbb{R})$ such that

$$(2) \qquad \exists u_- < a < u_+ \text{ in } \mathbb{R}: \quad f(u) \begin{cases} = 0 & \text{for } u \in \{u_-\,, a\,, u_+\}\,, \\ < 0 & \text{for } u \in (u_-, a)\,, \\ > 0 & \text{for } u \in (a, u_+)\,, \end{cases}$$

$$f'(u_-) < 0\,, \quad f'(u_+) < 0\,.$$

This kind of reaction-diffusion equation is known as Nagumo's equation to model propagation of signals [4,5], as one-dimensional real Ginzburg-Landau equation (RGLE) to model long-wave amplitudes e.g. in case of convection in binary mixtures near the onset of instability [6,7], as well as Allen-Cahn equation to model phase transitions in solids [8].

Following Allen and Cahn, a stable stationary state - such as u_- and u_+ - represents a phase of the system, whereas a traveling wave solution $u(x,t) = U(x-ct)$ with $\lim_{\xi\to\pm\infty} U(\xi) = u_\pm$ represents a phase transition. Each stationary state u_* has an associated potential $F(u_*) = F(u_-) + \int_{u_-}^{u_*} f(v)\ \mathrm{d}v$. One distinguishes between the balanced case, i.e. the stable states u_- and u_+ have the same potential $F(u_-) = F(u_+)$, and the unbalanced case, where the stable state with lesser potential value $F(u)$ is called the metastable state. Then a traveling wave solution $u(x,t) = U(x-ct)$ connecting the stable states u_- and u_+ will be stationary ($c = 0$) in the balanced case and moving in the direction of the metastable state in the unbalanced case.

In some applications it is important to include nonlocal effects. For example, Bates et al. [9] proposed a non-local model

$$(3) \qquad \tfrac{\partial u}{\partial t} = J * u - u + f(u) \quad \text{for } (x,t) \in \mathbb{R} \times (0,T]\,,$$

for even, non-negative functions $J \in C^1(\mathbb{R})$ with

$$\int_{\mathbb{R}} J(y)\ \mathrm{d}y = 1\,, \quad \int_{\mathbb{R}} |y| J(y)\ \mathrm{d}y < \infty\,, \quad J' \in L^1(\mathbb{R})\,,$$

and bistable functions f. The assumptions on J ensure that the problem exhibits a maximum principle and a variational formulation. The existence of traveling wave solutions $u(x,t) = U(x-ct)$ is concluded from a homotopy of (3) to a classical reaction-diffusion model (1). Moreover the traveling wave again will move depending on the balance of the potential values of the stable states. In contrast, the asymptotic stability is established only for stationary traveling wave solutions, i.e. in the balanced case, where an additional variational structure is available.

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Chen established a unified approach [10] to prove the existence, uniqueness and asymptotic stability with exponential decay of traveling wave solutions for the previous reaction-diffusion equations and many more examples from the literature. He considers general nonlinear nonlocal evolution equations in the form

$$\tfrac{\partial u}{\partial t}(x,t) = \mathcal{A}[u(\cdot,t)](x) \quad \text{for } (x,t) \in \mathbb{R} \times (0,T]\,,$$

where the nonlinear operator $\mathcal{A}$ is assumed to

1. be independent of t;
2. generate a L^∞ semigroup;
3. be translational invariant, i.e. $\mathcal{A}$ satisfies for all $u \in \operatorname{dom}\mathcal{A}$ the identity

$$\mathcal{A}[u(\cdot + h)](x) = \mathcal{A}[u(\cdot)](x+h) \quad \forall x\,, h \in \mathbb{R}\,.$$

   Consequently, there exists a function $f : \mathbb{R} \to \mathbb{R}$ which is defined by $\mathcal{A}[\alpha \mathbf{1}] = f(\alpha)\mathbf{1}$ for $\alpha \in \mathbb{R}$ and the constant function $\mathbf{1} : \mathbb{R} \to \mathbb{R}$, $x \mapsto 1$. This function f is assumed to be bistable (2);
4. satisfy a comparison principle

   If $\frac{\partial u}{\partial t} \geq \mathcal{A}[u]$, $\frac{\partial v}{\partial t} \leq \mathcal{A}[v]$ and $u(\cdot,0) \gneqq v(\cdot,0)$, then $u(\cdot,t) > v(\cdot,t)$ for all $t > 0$.

Chen's approach relies on the comparison principle and the construction of sub- and supersolutions for any given traveling wave solution. Importantly, the method does not depend on the balance of the potential.

At the same time, Zanette [11] proposed a model

$$\tfrac{\partial u}{\partial t} = D_0^\alpha u + f(u) \quad \text{for } (x,t) \in \mathbb{R} \times (0,T]\,, \tag{4}$$

with a fractional Laplacian D_0^α for some $\alpha \in (0,2)$ and an explicit bistable function f. This model exhibits monotone traveling wave solutions having an explicit integral representation, hence the asymptotic behavior of front tails and the front width can be studied directly. Subsequently, the reaction-diffusion equation (4) with fractional Laplacian and general bistable function f has been studied in the literature [11–18].

Engler [19] was one of the first to consider the scalar partial integro-differential equations

$$\tfrac{\partial u}{\partial t} = D_\theta^\alpha u + f(u) \quad \text{for } (x,t) \in \mathbb{R} \times (0,T]\,, \tag{5}$$

where $u = u(x,t)$, $f \in C^1(\mathbb{R})$ is a (bistable) nonlinear function, and D_θ^α is a Riesz-Feller operator with $1 < \alpha \leq 2$ and $|\theta| \leq \min\{\alpha, 2-\alpha\}$. A Riesz-Feller operator D_θ^α of order α and skewness θ can be defined as a

Fourier multiplier operator, see also the exposition of Mainardi, Luchko and Pagnini [20]. Starting from the fundamental solution of $\frac{\partial u}{\partial t} = D^\alpha_\theta u$, Engler constructs traveling wave solutions for some appropriate bistable function f. Assuming the existence of traveling wave solutions for general functions f, Engler studies the finiteness of the wave speed. The existence, uniqueness (up to translations), and stability of traveling wave solutions for general bistable functions is left open.

1.1. *Main analytical result.*

Our main result is summarized in the following theorem.

Theorem 1.1 ([21]). *Suppose* $1 < \alpha \le 2$, $|\theta| \le \min\{\alpha, 2-\alpha\}$ *and* $f \in C^\infty(\mathbb{R})$ *satisfies* (2). *Then equation* (5) *admits a traveling wave solution* $u(x,t) = U(x - ct)$ *satisfying*

$$\lim_{\xi \to \pm\infty} U(\xi) = u_\pm \quad \text{and } U'(\xi) > 0 \quad \text{for all } \xi \in \mathbb{R}\,. \tag{6}$$

In addition, a traveling wave solution of (5) *is unique up to translations. Furthermore, traveling wave solutions are globally asymptotically stable in the sense that there exists a positive constant* κ *such that if* $u(x,t)$ *is a solution of* (5) *with initial datum* $u_0 \in C_b(\mathbb{R})$ *satisfying* $0 \le u_0 \le 1$ *and*

$$\liminf_{x\to\infty} u_0(x) > a\,, \qquad \limsup_{x\to-\infty} u_0(x) < a\,, \tag{7}$$

then, for some constants ξ *and* K *depending on* u_0,

$$\|u(\cdot,t) - U(\cdot - ct + \xi)\|_{L^\infty(\mathbb{R})} \le K e^{-\kappa t} \qquad \forall t \ge 0\,.$$

1.2. *Discussion.*

To our knowledge, we established the first result [21] on existence, uniqueness (up to translations) and stability of traveling wave solutions of (5) with Riesz-Feller operators D^α_θ for $1 < \alpha < 2$ and $|\theta| \le \min\{\alpha, 2-\alpha\}$ and bistable functions f satisfying (2). The technical details of the proof are contained in [21], whereas in this paper we give a concise overview of the proof strategy and visualize the results also numerically.

To prove Theorem 1.1, we follow - up to some modifications - the approach of Chen [10]. His approach relies on the comparison principle and the construction of sub- and supersolutions for any given traveling wave solution. It allows to cover all bistable functions f satisfying (2) regardless of the balance of the potential and all Riesz-Feller operators D^α_θ for $1 < \alpha < 2$ regardless of $|\theta| \le \min\{\alpha, 2-\alpha\}$.

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Next, we quickly review different methods to study the traveling wave problem of a reaction-diffusion equation. In case of a classical reaction-local diffusion equation (1), the existence of traveling wave solutions can be studied via phase-plane analysis [1,22]. This method has no obvious generalization to our traveling wave problem for (5), since its traveling wave equation is an integro-differential equation.

The variational approach has been focused - so far - on symmetric diffusion operators such as fractional Laplacians and on balanced potentials, hence covering only stationary traveling waves [14–16,23]. The homotopy to a simpler traveling wave problem has been used to prove the existence of traveling wave solutions in case of (3), and (4) with unbalanced potential [17].

Chmaj [18] also considers the traveling wave problem for (4) with general bistable functions f. He approximates a given fractional Laplacian by a family of operators $J_\epsilon * u - (\int J_\epsilon)u$ such that $\lim_{\epsilon\to 0} J_\epsilon * u - (\int J_\epsilon)u = D_0^\alpha u$ in an appropriate sense. This allows him to obtain a traveling wave solution of (4) with general bistable function f as the limit of the traveling wave solutions u_ϵ of (3) associated to $(J_\epsilon)_{\epsilon\geq 0}$. It might be possible to modify Chmaj's approach to study also our reaction-diffusion equation (5) with Riesz-Feller operators. This would give an alternative existence proof of a traveling wave solutions.

However, Chen's approach allows to establish uniqueness (up to translations) and stability of traveling wave solutions as well. It remains an open problem to extend Chen's approach, if this is possible, to the general case of Riesz-Feller operators with $0 < \alpha \leq 1$ and $|\theta| \leq \min\{\alpha, 2-\alpha\}$.

1.3. *Outline.*

Our article is structured as follows. In Section 2, we give a non-technical review of our analytical results in a companion article [21]. We introduce the Riesz-Feller operators as Fourier multiplier operators on Schwartz functions, and extend the Riesz-Feller operators in form of singular integrals to functions in $C_b^2(\mathbb{R})$. The Riesz-Feller operators D_θ^α generate a convolution semigroup which we deduce from the theory of Lévy processes.

Then we present the analysis of the Cauchy problem for (5) with initial datum $u_0 \in C_b(\mathbb{R})$ such that $0 \leq u_0 \leq 1$. The proof follows a standard approach, to consider the Cauchy problem in its mild formulation and to prove the existence of a mild solution. The Cauchy problem generates a nonlinear semigroup which allows to prove uniform C_b^k estimates via a bootstrap argument and to conclude that mild solutions are also classical solutions.

A comparison principle is essential to prove our result on the existence, uniqueness and stability of traveling wave solutions and to allow for a larger class of admissible functions f in the result for the Cauchy problem.

Finally, we consider the traveling wave problem for (5). In [21] we consider a general approach by Chen [10]. There we study his necessary assumptions and notice that some estimates are not of the required form. However Chen's approach can be extended, which we prove in [21, Appendices A–C]. We sketch the proof of Theorem 1.1 in Section 2, and refer to [21, Subsection 4.2] for more details.

In Section 3, we review numerical methods for reaction-diffusion equations with fractional Laplacian and discuss the (im-)possibility of extensions to our reaction-diffusion equations with Riesz-Feller operators. Then we present a direct method using integral operator discretization based on quadrature in combination with projection boundary conditions. Furthermore, we visualize the analytical results from Section 2 and outline several challenges for the numerical analysis of asymmetric Riesz-Feller operators.

2. Traveling wave solutions.

A Riesz-Feller operator of order α and skewness θ can be defined as a Fourier multiplier operator,

$$\mathcal{F}[D^\alpha_\theta f](\xi) = \psi^\alpha_\theta(\xi)\mathcal{F}[f](\xi)\,, \qquad \xi \in \mathbb{R}\,, \tag{8}$$

with symbol

$$\psi^\alpha_\theta(\xi) = -|\xi|^\alpha \exp\left[i(\operatorname{sgn}(\xi))\theta \tfrac{\pi}{2}\right]\,, \tag{9}$$

for some $0 < \alpha \leq 2$ and $|\theta| \leq \min\{\alpha, 2-\alpha\}$. The symbol $\psi^\alpha_\theta(\xi)$ is the logarithm of the characteristic function of a Lévy strictly stable probability density with index of stability α and asymmetry parameter θ according to Feller's parameterization [24,25].

Remark 2.1. We follow the convention in probability theory and define the Fourier transform of f in the Schwartz space $\mathcal{S}(\mathbb{R})$ as

$$\mathcal{F}[f](\xi) := \int_{\mathbb{R}} e^{+i\xi x} f(x)\ \mathrm{d}x\,, \qquad \xi \in \mathbb{R}\,,$$

and the inverse Fourier transform as

$$\mathcal{F}^{-1}[f](x) := \tfrac{1}{2\pi} \int_{\mathbb{R}} e^{-i\xi x} f(\xi)\ \mathrm{d}\xi\,, \qquad x \in \mathbb{R}\,.$$

Moreover, $\mathcal{F}$ and $\mathcal{F}^{-1}$ will denote also their respective extensions to $L^2(\mathbb{R})$.

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To analyze the Cauchy problem for the reaction diffusion equation (5) we need to investigate the linear space-fractional diffusion equation

$$\frac{\partial u}{\partial t}(x,t) = D^\alpha_\theta[u(\cdot,t)](x) \quad \text{for } (x,t) \in \mathbb{R}\times(0,\infty)\,, \tag{10}$$

$0 < \alpha \le 2$ and $|\theta| \le \min\{\alpha, 2-\alpha\}$. A formal Fourier transform of the associated Cauchy problem yields

$$\frac{\partial}{\partial t}\mathcal{F}[u](\xi,t) = \psi^\alpha_\theta(\xi)\mathcal{F}[u](\xi,t)\,, \qquad \mathcal{F}[u](\xi,0) = \mathcal{F}[u_0](\xi)\,,$$

which has a solution $\mathcal{F}[u](\xi,t) = e^{t\psi^\alpha_\theta(\xi)}\mathcal{F}[u_0](\xi)$. Hence, a formal solution of the Cauchy problem is given by

$$u(x,t) = (G^\alpha_\theta(\cdot,t) * u_0)(x) \tag{11}$$

with kernel (or Green's function) $G^\alpha_\theta(x,t) := \mathcal{F}^{-1}\left[\exp(t\psi^\alpha_\theta(\cdot))\right](x)$.

Due to Theorem [26, Theorem 14.19], the function $e^{t\psi^\alpha_\theta(\xi)}$ is the characteristic function of a random variable with Lévy strictly α-stable distribution. Thus G^α_θ is the scaled probability measure of a Lévy strictly α-stable distribution. In case of $(\alpha,\theta) \in \{(0,0),(1,1),(1,-1)\}$, the probability measure G^α_θ is a delta distribution

$$G^0_0(x,t) = \delta_x\,, \qquad G^1_1(x,t) = \delta_{x+t}\,, \qquad G^1_{-1}(x,t) = \delta_{x-t}$$

and called trivial [26, Definition 13.6]. In all other (non-trivial) cases, the probability measure G^α_θ is absolutely continuous with respect to the Lebesgue measure and has a continuous probability density [26, Proposition 28.1], which we will denote again by G^α_θ. For every infinitely divisible distribution μ on $\mathbb{R}^d$, such as G^α_θ, there exists an associated Lévy process $(X_t)_{t\ge0}$. In particular, every Lévy process exhibits an associated strongly continuous semigroup on $C_0(\mathbb{R}^d)$, see also [26, Theorem 31.5].

The infinitesimal generator of our Lévy process has the following representation, which allows to extend the Riesz-Feller operator to $C^2_b(\mathbb{R})$-functions.

Theorem 2.1. *If $0 < \alpha < 1$ or $1 < \alpha < 2$ and $|\theta| \le \min\{\alpha, 2-\alpha\}$, then for all $f \in \mathcal{S}(\mathbb{R})$ and $x \in \mathbb{R}$*

$$D^\alpha_\theta f(x) = \frac{c_1-c_2}{1-\alpha} f'(x) + c_1 \int_0^\infty \frac{f(x+\xi)-f(x)-f'(x)\,\xi\,\mathbf{1}_{(-1,1)}(\xi)}{\xi^{1+\alpha}}\,\mathrm{d}\xi + c_2 \int_0^\infty \frac{f(x-\xi)-f(x)+f'(x)\,\xi\,\mathbf{1}_{(-1,1)}(\xi)}{\xi^{1+\alpha}}\,\mathrm{d}\xi \tag{12}$$

where $\mathbf{1}_{(-1,1)}(\cdot)$ is an indicator function and some constants $c_1, c_2 \ge 0$ with $c_1 + c_2 > 0$.

Proof. The result follows from [26, Theorem 31.7] see also [21, Theorem 2.4]. □

In the analysis of the traveling wave problem, we are mostly interested in the evolution of initial data in C_b. Therefore, it is important to notice the following proposition.

Proposition 2.1 ([21, Corollary 2.10]). *For $1 < \alpha < 2$ and $|\theta| \leq \min\{\alpha, 2-\alpha\}$, the Riesz-Feller operator D^α_θ generates a convolution semigroup $S_t : C_b(\mathbb{R}) \to C_b(\mathbb{R})$, $u_0 \mapsto S_t u_0 = G^\alpha_\theta(\cdot, t) * u_0$, with kernel $G^\alpha_\theta(x,t)$. Moreover, the convolution semigroup with $u(x,t) := S_t u_0$ satisfies*

1. *$u \in C^\infty(\mathbb{R} \times (t_0, \infty))$ for all $t_0 > 0$;*
2. *$\frac{\partial u}{\partial t} = D^\alpha_\theta u$ for all $(x,t) \in \mathbb{R} \times (t_0, \infty)$ and any $t_0 > 0$;*
3. *If $u_0 \in C_b(\mathbb{R})$ then $u \in C_b(\mathbb{R} \times [0,T])$ for any $T > 0$.*

This result states that Riesz-Feller operators D^α_θ for $0 < \alpha \leq 2$ and $|\theta| \leq \min\{\alpha, 2-\alpha\}$ generate conservative C_b-Feller semigroups. This can be deduced from a criterion on the symbol of Fourier multiplier operators in [27].

It is important to notice that $S_t : C_b(\mathbb{R}) \to C_b(\mathbb{R})$ is not a strongly continuous semigroup. Thus the C^2_b-extension of D^α_θ are not the infinitesimal generators of the C_b-extension of the strongly continuous semigroup $(S_t)_{t\geq 0}$ on $C_0(\mathbb{R})$ in the usual sense.

2.1. *Cauchy problem.*

We consider the Cauchy problem

$$\begin{cases} \frac{\partial u}{\partial t} = D^\alpha_\theta u + f(u) & \text{for } (x,t) \in \mathbb{R} \times (0,\infty)\,, \\ u(x,0) = u_0(x) & \text{for } x \in \mathbb{R}\,, \end{cases} \tag{13}$$

for $1 < \alpha \leq 2$, $|\theta| \leq \min\{\alpha, 2-\alpha\}$ and $f \in C^\infty(\mathbb{R})$ satisfying (2). We follow a standard approach, and consider the Cauchy problem in its mild formulation to prove the existence of a mild solution. The Cauchy problem generates a nonlinear semigroup which allows to prove uniform C^k_b estimates via a bootstrap argument and to conclude that mild solutions are also classical solutions.

Theorem 2.2 ([21, Theorem 3.3]). *Suppose $1 < \alpha \leq 2$, $|\theta| \leq \min\{\alpha, 2-\alpha\}$ and $f \in C^\infty(\mathbb{R})$ satisfies (2). The Cauchy problem (5) with initial condition $u(\cdot, 0) = u_0 \in C_b(\mathbb{R})$ and $0 \leq u_0 \leq 1$ has a solution $u(x,t)$ in the following sense: for all $T > 0$*

1. *$u \in C_b(\mathbb{R} \times (0,T))$ and $u \in C^\infty_b(\mathbb{R} \times (t_0, T))$ for all $t_0 \in (0,T)$;*

2. *u satisfies* (5) *on* $\mathbb{R} \times (0,T)$*;*
3. *If* $u_0 \in C_b(\mathbb{R})$ *then* $u(\cdot,t) \to u_0$ *uniformly as* $t \to 0$*;*
4. $0 \le u(x,t) \le 1$ *for all* $(x,t) \in \mathbb{R} \times (0,\infty)$*;*
5. $\forall k \in \mathbb{N}\ \forall t_0 > 0\ \exists C > 0$ *such that* $\|u(\cdot,t)\|_{C_b^k(\mathbb{R})} \le C\ \forall 0 < t_0 < t$.

The following comparison principle is essential to prove our result on the existence, uniqueness and stability of traveling wave solutions and to allow for a larger class of admissible functions f in the result for the Cauchy problem.

Lemma 2.1 ([21, Lemma 3.4]). *Assume* $1 < \alpha \le 2$, $|\theta| \le \min\{\alpha, 2-\alpha\}$, $T > 0$ *and* $u, v \in C_b(\mathbb{R}\times[0,T]) \cap C_b^2(\mathbb{R}\times(t_0,T])$ *for all* $t_0 \in (0,T)$ *such that*

$$\tfrac{\partial u}{\partial t} \le D_\theta^\alpha u + f(u) \quad \text{and} \quad \tfrac{\partial v}{\partial t} \ge D_\theta^\alpha v + f(v) \quad \text{in } \mathbb{R}\times(0,T]\,.$$

1. *If* $v(\cdot,0) \ge u(\cdot,0)$ *then* $v(x,t) \ge u(x,t)$ *for all* $(x,t) \in \mathbb{R}\times(0,T]$.
2. *If* $v(\cdot,0) \gneqq u(\cdot,0)$ *then* $v(x,t) > u(x,t)$ *for all* $(x,t) \in \mathbb{R}\times(0,T]$.
3. *Moreover, there exists a positive continuous function*

$$\eta : [0,\infty)\times(0,\infty) \to (0,\infty)\,, \quad (m,t) \mapsto \eta(m,t)\,,$$

such that if $v(\cdot,0) \ge u(\cdot,0)$ *then for all* $(x,t) \in \mathbb{R}\times(0,T)$

$$v(x,t) - u(x,t) \ge \eta(|x|,t) \int_0^1 v(y,0) - u(y,0)\ \mathrm{d}y\ .$$

Sketch of the proof of Theorem 1.1. We present here a sketch of the proof of Theorem 1.1 and refer to our article [21] for more details. To prove existence of traveling wave solutions satisfying (6), we consider the Cauchy problem for (5) with some smooth initial datum $u_0 \in C_b(\mathbb{R})$ satisfying (6). Due to Theorem 2.2 there exists a classical solution $u(x,t)$. We consider a diverging sequence $\{t_j\}_{j\in\mathbb{N}}$ such that $\lim_{j\to\infty} t_j = \infty$ and the associated sequence $\{u(\cdot,t_j)\}_{j\in\mathbb{N}}$ in $C_b(\mathbb{R})$. Then, due to Arzela-Ascoli Theorem, there exists a subsequence and a limiting function $\tilde{u}$ such that $\lim_{k\to\infty} u(\cdot,t_{j_k}) = \tilde{u}(\cdot)$. The final and most important step is to verify that $\tilde{u}$ is a traveling wave solution of (5) satisfying (6).

To prove uniqueness (up to translations) of a traveling wave solution, sub- and super-solutions of (5) are constructed from any given traveling wave solution. Assuming the existence of two traveling wave solutions, one traveling wave solution is bounded from below and from above by suitable sub- and super-solutions associated to the other traveling wave solution, respectively. The comparison principle in Lemma 2.1 allows to show that one traveling wave solution is a translated version of the other traveling wave solution.

To prove stability of a traveling wave solution, considering the Cauchy problem for (5) with initial datum u_0 satisfying (7), then the associated solution v can be bounded from below and from above by suitable sub- and super-solutions associated to the traveling wave solution, respectively. The comparison principle and the evolution of sub- and super-solutions show that these bounds on the solution v get tighter and allow to prove the exponential convergence to (a translated version of) the traveling wave solution.

For more details see the proof of [21, Theorem 4.6]. □

3. Numerical methods.

In this section, we illustrate our results from Theorem 1.1 and discuss numerical methods for (5). The case $\theta = 0$ yields the fractional Laplacian $D_0^\alpha = -(-\Delta)^{\alpha/2}$ which has been discussed frequently from a numerical perspective in the literature. Hence, there is a notational convention to write (5) for $\theta = 0$ as

$$\frac{\partial u}{\partial t} + (-\Delta)^{\alpha/2} = f(u) \qquad \text{or} \qquad \frac{\partial u}{\partial t} = -(-\Delta)^{\alpha/2} + f(u).$$

However, we shall adhere to the convention D_0^α as introduced previously. First, we review some of the available numerical schemes for this case. We restrict the computational domain from $x \in \mathbb{R}$ to $x \in [-b, b] =: \Omega$ for some (sufficiently large) $b > 0$ and with Neumann or Dirichlet boundary conditions. A numerical comparison of various methods for the case D_0^α has already been carried out in [28,29] so we shall focus our small survey in Sections 3.1-3.4 on the difficulties in the numerical generalization from $\theta = 0$ to $\theta \neq 0$ for space-fractional equations. Furthermore, we only cover spatial grid bases schemes and do not discuss stochastic particle methods.

The main novel results are our direct method using integral operator discretization in combination with projection boundary conditions in Section 3.5 and the numerical results in Section 3.6 for (5).

3.1. *Spectral methods.*

One idea is to generalize *spectral methods* to the fractional Laplacian case [30]. Let λ_j denote the Laplacian eigenvalues and ϕ_j the corresponding eigenfunctions for $D_0^2\phi_l = \lambda_l\phi_l$ with $l \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Consider $L^2(\Omega)$ then

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we may write $v \in L^2(\Omega)$ as a series expansion

$$v = \sum_{l=0}^{\infty} \hat{v}_l \phi_l, \qquad \hat{v}_l := \langle v, \phi_l \rangle \tag{14}$$

where $\langle \cdot, \cdot \rangle$ denotes the $L^2(\Omega)$ inner product. Fix some α with $1 < \alpha \leq 2$ and consider

$$H^{\alpha/2}(\Omega) := \left\{ v \in L^2(\Omega) : \sum_{l=0}^{\infty} |\hat{v}_l|^2 |\lambda_l|^{\alpha/2} < \infty \right\}. \tag{15}$$

The spectral decomposition of the fractional Laplacian implies [31] that $-(-\lambda_l)^{\alpha/2}$ are eigenvalues with eigenfunctions ϕ_l for D_0^α and for any $v \in H^{\alpha/2}(\Omega)$ we have

$$D_0^\alpha v = -\sum_{l=0}^{\infty} (-\lambda_l)^{\alpha/2} \hat{v}_l \phi_l. \tag{16}$$

As a remark, we note that all the minus signs on the right-hand side in (16) disappear if we would write $(-\Delta)^{\alpha/2} u$ on the left-hand side instead and would let λ_l denote the eigenvalues of the negative Laplacian. It is suggested in [30] to apply a backward Euler-type time discretization on a mesh

$$0 = t_0 < t_1 < \cdots < t_m < t_{m+1} < \cdots < T \tag{17}$$

for (5) where we set $t_{m+1} - t_m =: (\delta t)_m$. Denote by $u^m := u(x, t_m)$ the solution at time t_m. For the time step t_m to t_{m+1} one may consider the semi-implicit backward Euler scheme

$$\frac{u^{m+1} - u^m}{(\delta t)_m} = D_0^\alpha u^{m+1} + f(u^m). \tag{18}$$

Making the Fourier spectral ansatz

$$u(x,t) = \sum_{l=0}^{\infty} \hat{u}_l(t) \phi_l(x) \approx \sum_{l=0}^{L} \hat{u}_l(t) \phi_l(x)$$

in (18), using the orthogonality of the basis functions ϕ_l and employing (16) leads to the numerical method

$$\hat{u}_l^{m+1} = \frac{1}{1 + (-\lambda_l)^{\alpha/2} (\delta t)_m} \left(\hat{u}_l^m + (\delta t)_m \hat{f}_l(u^m) \right) \tag{19}$$

where $\hat{f}_l$ is the l-th Fourier coefficient of f. In particular, the $L+1$ Fourier modes in (19) are decoupled and relatively easy to solve for. Further implementation details of (19) can be found in [30, Code 4, p.10]. However, the generalization of (19) from the fractional Laplacian case D_0^α to the asymmetric case D_θ^α with $\theta \neq 0$ is not straightforward. In fact, in the asymmetric case one generically obtains complex eigenvalues and a continuous spectrum [32]. This means that (16) is no longer valid for $\theta \neq 0$. For another approach using transform/Fourier-type techniques we refer to [33].

3.2. *Finite difference methods.*

A second possible approach to solve (5) is to use a *finite difference method* (FDM) [34] combined with the Grünwald-Letnikov representation of the space fractional derivative. Let us consider a spatial discretization of $\Omega = [-b, b]$ as follows

$$-b = x_1 < x_2 < \cdots < x_N = b. \tag{20}$$

We still use the temporal discretization (17). For D_0^α the Grünwald-Letnikov representation of D_0^α is given by

$$\begin{aligned}(D_0^\alpha u)(x,t) &= \lim_{N\to\infty} \frac{1}{h_+^\alpha} \sum_{r=0}^{N} \frac{\Gamma(r-\alpha)}{\Gamma(-\alpha)\Gamma(r+1)} u(x - rh_+, t) \\ &\quad + \lim_{N\to\infty} \frac{1}{h_-^\alpha} \sum_{r=0}^{N} \frac{\Gamma(r-\alpha)}{\Gamma(-\alpha)\Gamma(r+1)} u(x + rh_+, t),\end{aligned} \tag{21}$$

where $h_+ = (x+b)/N$ and $h_- = (b-x)/N$. Let us assume for simplicity that the spatial grid is equidistant and let $h := 2b/N$. Furthermore, we let

$$u_n^m := u(x_n, t_m).$$

Then one possible finite-difference discretization of (5) is given by [28,34]

$$\frac{u_n^{m+1} - u_n^m}{(\delta t)_m} = \frac{1}{h^\alpha} \left[\sum_{r=0}^{n+1} g_r u_{n-r+1}^{m+1} + \sum_{r=0}^{N-n+1} g_r u_{n+r-1}^{m+1} \right] + f(u_n^{m+1}),$$

with $g_r := \frac{\Gamma(r-\alpha)}{\Gamma(-\alpha)\Gamma(r+1)}$. For a similar approach using the Grünwald-Letnikov representation to obtain finite-difference schemes we also refer to [35–42]. For even more details on finite-difference methods for space-fractional diffusion equations consider [43–45]. In some sense, our scheme in Section 3.5 has an analogous starting point. However, instead of the Grünwald-Letnikov representation we use the integral representation formula which we also employed in the existence-uniqueness-stability proof of Theorem 1.1; see also Section 3.5.

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3.3. *Finite element methods.*

Another quite natural possibility is to follow the classical *finite element method* (FEM) variational approach. We follow [46,47] in our exposition for the case $\theta = 0$. Let $X := H_0^{\alpha/2}(\Omega)$ denote the usual fractional Sobolev space obtained as a closure of $C_0^\infty(\Omega)$ in $H^{\alpha/2}(\Omega)$ and define

$$A(v, w) := -\langle D_0^{\alpha/2} v, D_0^{\alpha/2} w\rangle\,, \tag{22}$$

where representation (16) is used. Then one may check that A is coercive and continuous. Consider the space X_h of piecewise linear continuous functions in X with compact support given by

$$X_h := \{v \in C_0(\Omega) : v \text{ is linear over } [x_n, x_{n+1}],\ n = 1, 2, \ldots, N-1\}.$$

Then we may define a discrete operator $A_h : X_h \to X_h$ associated to A via

$$\langle A_h v_h, w_h\rangle = A(v_h, w_h) \qquad \forall v_h, w_h \in X_h.$$

A semi-discrete Galerkin FEM scheme for (5) is to find $u_h = u_h(t) \in X_h$ such that

$$\left\langle \frac{\partial u_h}{\partial t}(t), v_h \right\rangle = \langle A_h u_h(t), v_h\rangle + \langle f(u_h(t)), v_h\rangle \qquad \forall v_h \in X\,, \tag{23}$$

and projected initial condition $\langle u_h(0), v_h\rangle = \langle u(0), v_h\rangle$. Choosing a basis $\{\varphi_1, \varphi_2, \ldots, \varphi_N\}$ of X_h we may write

$$u_h(x, t) = \sum_{n=1}^{N} u_n(t)\varphi_n(x).$$

One defines the usual mass matrix $M \in \mathbb{R}^{N\times N}$ and stiffness matrix $A \in \mathbb{R}^{N\times N}$ with entries

$$M_{nm} = \langle \varphi_n, \varphi_m\rangle, \qquad A_{nm} = A(\varphi_n, \varphi_m), \qquad m, n \in \{1, 2, \ldots, N\}. \tag{24}$$

This converts (23) into the ODEs

$$M\frac{dU}{dt} = AU + f(U) \tag{25}$$

where $U = (u_1, \ldots, u_N)^T$ and $f(U) = (\langle f(u_h), \varphi_1\rangle, \ldots, \langle f(u_h), \varphi_N\rangle)^T$. Then one may use a time-stepping scheme directly. For example, a backward Euler semi-implicit discretization yields for $U^m := U(t_m)$ the method

$$(\mathrm{Id} - (\delta t)_m M^{-1} A)U^{m+1} = U^m + (\delta t)_m M^{-1} f(U^m). \tag{26}$$

These considerations show that we can, at least formally, just follow the classical FEM theory to derive numerical methods for equations involving D_0^α. However, for the fractional Laplacian D_0^α the matrix entries for A defined in (24) are not as easy to compute as for D_0^2. FEM techniques also seem to generalize formally to the asymmetric case as coercivity and continuity hold for classes of fractional operators more general than D_0^α [46, p.574-575]. However, we are again faced with the practical problem of computing (an approximation of) $A(\varphi_n, \varphi_m)$. This observation is one reason which motivates the method presented in the next section. For more details on FEM for space fractional equations we refer to [48–51].

3.4. *Matrix-transfer techniques.*

The symmetry of D_0^2 and the view of fractional powers D_0^α can be employed in conjunction with FEM or FDM discretizations for (5). Again, we consider the case $\theta = 0$ following [52–54]. Let $A_\Delta \in \mathbb{R}^{N\times N}$ be the usual FEM stiffness matrix and M_Δ be the FEM mass matrix for D_0^2. One natural idea is to use a fractional power of the matrix $B_\Delta := M_\Delta^{-1} A_\Delta$ in a numerical scheme to represent the fractional Laplacian. Suppose we can compute $(B_\Delta)^\alpha$ then a backward semi-implicit Euler-type time discretization, similar to (26), leads to

$$(\mathrm{Id} - (\delta t)_m (B_\Delta)^\alpha) U^{m+1} = U^m + M_\Delta^{-1} f(U^m). \tag{27}$$

To solve (27) one has to also compute the function

$$q(z) = \frac{1}{1 - (\delta t)_m z^\alpha} \tag{28}$$

efficiently for matrices, which has been discussed in [54]. However, we still have to define B_Δ^α. Standard theory implies that A_Δ, M_Δ are real, symmetric matrices [55]. Furthermore, A_Δ is non-negative definite and M_Δ is positive definite. A direct calculation shows that

$$(M_\Delta)^{1/2} B_\Delta (M_\Delta)^{-1/2} = (M_\Delta)^{-1/2} A_\Delta (M_\Delta)^{-1/2}.$$

Therefore, B_Δ is similar to a real, symmetric matrix with well-defined point spectrum $\sigma(B_\Delta) \subset \mathbb{R}$ and eigenvalues $\xi_1 \leq \xi_2 \leq \cdots \leq \cdots \xi_N$. Then it is very natural to define a matrix function $q(Z)$, including (28) as a special case, by

$$q(Z) = Q q(\Xi) Q^{-1},$$

where Ξ is a diagonal matrix with $\Xi_{nn} = \xi_n$, Q consists of the eigenvectors associated to the eigenvalues ξ_n and $[q(\Xi)]_{nn} = q(\xi_n)$. This yields a well-defined fractional power $(B_\Delta)^\alpha$ when applied to $q(z) = z^\alpha$ and can then also be applied to define (28). Unfortunately, the matrix transfer technique does not generalize immediately to the case $\theta \neq 0$ as the spectrum $\sigma(D^\alpha_\theta)$ for $\theta \neq 0$ is generically continuous with complex eigenvalues as already discussed in Section 3.1. For more on the matrix transfer technique we refer to [56,57].

3.5. *Integral representation, quadrature and projection boundary conditions.*

Sections 3.1-3.4 explain why, to the best of our knowledge, there seem to be very few (if any) detailed numerical studies of the asymmetric case $\theta \neq 0$ for the nonlinear Allen-Cahn/Nagumo-type Riesz-Feller reaction-diffusion equation (5).

Here we present an easy-to-implement method to study (5) numerically with a focus on the dynamics of traveling waves. Our approach is to use the integral representation of Riesz-Feller operators to view (5) as an integro-differential equation. For $\alpha \in (1, 2)$ the representation formula is given by [21]

$$
\begin{aligned}
(D^\alpha_\theta u)(x,t) = c_1 \int_0^\infty \frac{u(x+\xi,t) - u(x,t) - \xi \frac{\partial u}{\partial x}(x,t)}{\xi^{1+\alpha}} \, \mathrm{d}\xi \\
+ c_2 \int_0^\infty \frac{u(x-\xi,t) - u(x,t) + \xi \frac{\partial u}{\partial x}(x,t)}{\xi^{1+\alpha}} \, \mathrm{d}\xi
\end{aligned}
\tag{29}
$$

where the constants $c_{1,2}$ are given in [20] as

$$
c_1 = \frac{\Gamma(1+\alpha)\sin\left((\alpha+\theta)\frac{\pi}{2}\right)}{\pi} \quad \text{and} \quad c_2 = \frac{\Gamma(1+\alpha)\sin\left((\alpha-\theta)\frac{\pi}{2}\right)}{\pi}.
$$

Note that there is also an integral representation for $\alpha \in (0, 1)$ [21] for $x \in \mathbb{R}$. Furthermore, there is an analogous integral representation formula for fractional Laplacians in $\mathbb{R}^d$ in [58]. Hence, starting from a representation like (29) is not really a restriction, even for higher-dimensional cases. Furthermore, a similar strategy has also been applied successfully in a similar to context to other nonlocal operator equations [59] involving traveling waves. If we write

$$
\begin{aligned}
g_1(\xi,x,t) &:= \frac{u(x+\xi,t) - u(x,t) - \xi \frac{\partial u}{\partial x}(x,t)}{\xi^{1+\alpha}}, \\
g_2(\xi,x,t) &:= \frac{u(x-\xi,t) - u(x,t) + \xi \frac{\partial u}{\partial x}(x,t)}{\xi^{1+\alpha}},
\end{aligned}
$$

then we can simply re-write (5) as an integro-differential equation

$$
\frac{\partial u}{\partial t}(x,t) = c_1 \int_0^\infty g_1(\xi,x,t) \, \mathrm{d}\xi + c_2 \int_0^\infty g_2(\xi,x,t) \, \mathrm{d}\xi + f(u(x,t)). \tag{30}
$$

For simplicity, we shall just introduce our method for a uniform spatial mesh (20), i.e. we have

$$(31)\quad -b = x_1 < x_2 < \dots < x_N = b \qquad \text{with} \quad x_{n+1} - x_n = \frac{2b}{N-1} =: h,$$

where we assume that $N \geq 3$ is odd so that $x_{(N+1)/2} = 0$. Furthermore, we use another spatial mesh to approximate the integral operators (29) over a finite domain obtained as a sub-mesh from (31) as follows

$$(32)\qquad \xi_1 = x_{\frac{N+1}{2}+1}, \quad \xi_2 = x_{\frac{N+1}{2}+2}, \quad \dots, \quad \xi_{M+1} = x_N,$$

which has M subintervals $[\xi_m, \xi_{m+1}]$. We may easily relate M to the number of points N in our original mesh by $M = (N-1)/2$. For b, N sufficiently large we may just use a quadrature rule to approximate (29); we remark that the possibility to use quadrature techniques for time-fractional Caputo-derivative fractional equations has already been noticed in [60,61]. Very recently (in fact, during the preparation of this work), Huang and Oberman [62] proposed a quadrature-scheme based upon a singular integral presentation of the symmetric case D_0^α. We use the regularized, fully asymmetric representation (29) and obtain for the trapezoidal rule, with $\rho \in \{1,2\}$, that

$$\begin{aligned} c_\rho \int_0^\infty g_\rho(\xi, x, t)\, \mathrm{d}\xi &\approx c_\rho \int_h^b g_\rho(\xi, x, t)\, \mathrm{d}\xi \\ &\approx \frac{c_\rho(b-h)}{2M}\left[g_\rho(\xi_1, x, t) + g_\rho(\xi_{M+1}, x, t) + 2\sum_{m=2}^{M} g_\rho(\xi_m, x, t)\right]. \end{aligned}$$

Using this approximation in (30) yields a system of (formal) ODEs for $u_n(t) = u(x_n, t)$, which can be written as

$$(33)\quad \begin{aligned} \frac{\mathrm{d}u_n}{\mathrm{d}t} = \frac{b-h}{2M}\Bigg[& c_1\left((g_1(u))_{n,1} + (g_1(u))_{n,M+1}\right) + 2c_1 \sum_{m=2}^{M} (g_1(u))_{n,m} \\ & + c_2\left((g_2(u))_{n,1} + (g_2(u))_{n,M+1}\right) + 2c_2 \sum_{m=2}^{M} (g_2(u))_{n,m}\Bigg] + f(u_n) \end{aligned}$$

for $n \in \{1, 2, \dots, N\}$, where the terms involving $(g_\rho(u))_{n,m}$ are given by

$$(g_1(u))_{n,m} = \tfrac{u_{n+m} - u_n}{\xi_m^{1+\alpha}} - \tfrac{u_{n+1} - u_n}{\xi_m^\alpha h}, \qquad (g_2(u))_{n,m} = \tfrac{u_{n-m} - u_n}{\xi_m^{1+\alpha}} + \tfrac{u_{n+1} - u_n}{\xi_m^\alpha h}.$$

Of course, the system (33) is, as yet, only a formal representation as it involves spatial mesh indices for u which lie outside the range i.e. $u_n =$

$u(x_n, t)$ for $n \in \{1, 2, \ldots, N\}$. There is a choice of boundary conditions. However, instead of classical Neumann or Dirichlet conditions, we want to compute traveling waves which satisfy

$$\lim_{x \to -\infty} u(x,t) = u_-, \qquad \lim_{x \to +\infty} u(x,t) = u_+$$

for constants $u_\pm$. Hence, we adopt the following projection-type boundary conditions for the numerical method

$$u_n = \begin{cases} u_N & \text{if } n \geq N, \\ u_1 & \text{if } n \leq 1, \end{cases} \tag{34}$$

Using (34), we get a well-defined ODE system (33) which can be solved using forward integration, i.e. we adopt a method-of-lines approach; for more details on using projection boundary conditions to compute traveling waves in the classical FitzHugh-Nagumo equation we refer e.g. to [63–65].

Regarding our algorithm (33)-(34) for waves of the Riesz-Feller bistable equation, we emphasize that our approach is clearly non-optimal from a numerical perspective. For example, there are straightforward generalizations to non-uniform meshes and higher-order schemes by using non-uniform-mesh higher-order quadrature methods. We leave these generalizations as future challenges. Here, we are primarily interested in developing a simple scheme for (5) and to visualize some of the results from Theorem 1.1.

3.6. *Numerical results.*

In this section, we briefly discuss some numerical simulations of (5) with $f(u) = u(1-u)(u-a)$ for some $a \in (0,1)$ using our algorithm from Section 3.5. Unless stated otherwise, we fix $\Omega = [-b, b] = [-30, 30]$, $N = 181$, spatial mesh points and always employ a standard stiff ODE solver to solve (33)-(34) (more precisely, `ode15s` from MatLab [66]) for $t \in [0, T]$. Figure 1(a) shows the initial condition

$$u_0(x) = u(x,0) = \begin{cases} 0 & \text{if } x \in [-30, -2), \\ \frac{1}{4}x + \frac{1}{2} & \text{if } x \in [-2, 2], \\ 1 & \text{if } x \in (2, 30]. \end{cases} \tag{35}$$

The initial condition (35) is important as it has been used in the existence part of the proof of Theorem 1.1 as discussed in [21,67]. In particular, u_0 is shown to converge to a traveling wave.

Figure 1(b)-(c) show the fully asymmetric fractional case with D^α_θ for $\alpha = 1.8$ and $\theta = 0.1$. In both cases we observe a rapid smoothing effect of the solution as predicted by the smoothing result in Theorem 1.1. Furthermore,

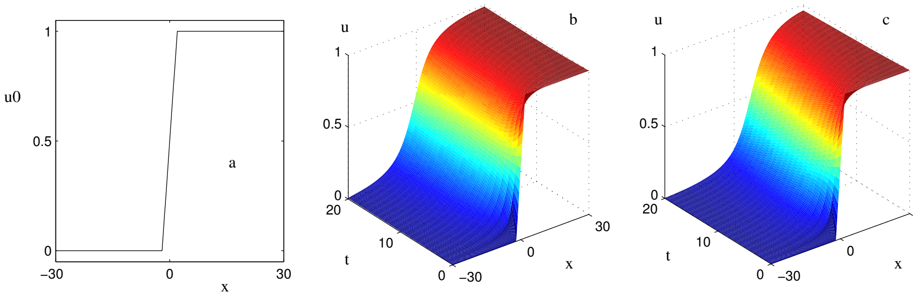

Figure 1. Fixed parameter values are $\theta = 0.1$, $\alpha = 1.8$, $T = 20$. (a) Initial condition $u_0 = u(x, 0)$ given by (35). (b) Simulation with $a = 0.5$. (c) Simulation with $a = 0.6$, the wave travels to the right.

in both cases, convergence to a traveling wave profile is observed, where moving the parameter a changes the wave speed. Again, this is expected since the supremum-norm of the nonlinearity $f(u) = u(1-u)(u-a)$ does influence the wave speed.

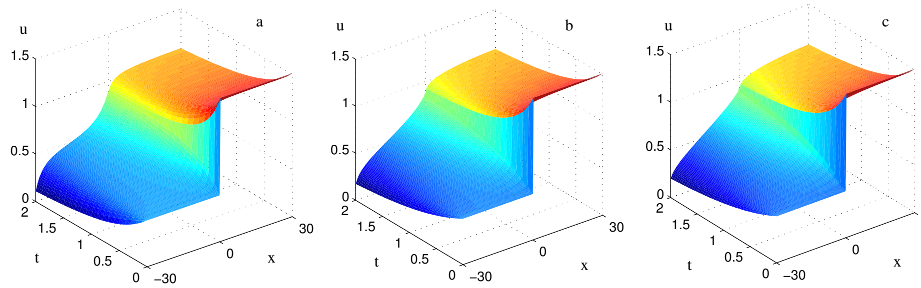

Figure 2. Fixed parameter values are $\theta = 0.1$, $a = 0.5$, $T = 2$ with initial condition (36). (a) $\alpha = 1.8$. (b) $\alpha = 1.2$. (c) $\alpha = 1.01$.

As a second interesting part we are interested in discontinuous initial conditions bounded away from the traveling wave, and even violating one of the stability assumptions ($0 \le u_0 \le 1$) from Theorem 1.1. One example is

$$u_0(x) = u(x,0) = \begin{cases} 0.49 \text{ if } x \in [-30, 0], \\ 1.51 \text{ if } x \in (0, 30]. \end{cases} \tag{36}$$

Furthermore, we vary the fractional exponent α. Figure 2 shows the results. Although the initial condition is not within the framework of the theoretical analysis, we still observe extremely rapid convergence to a wave profile where the end-states move to $u(-b, t) = 0$ and $u(b, t) = 1$. Note however, that the convergence, as well as the regularization effect, seems to be slower for smaller exponents α.

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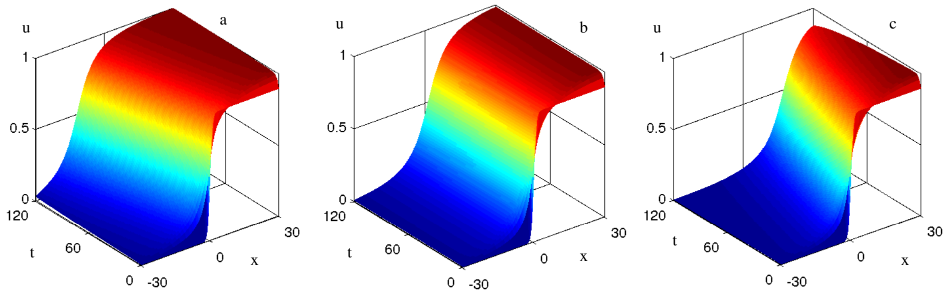

Figure 3. Fixed parameter values are $\alpha = 1.5$, $a = 0.5$, $T = 120$ with initial condition (35). (a) $\theta = 0.2$; wave moves to the left. (b) $\theta = 0.0$; standing wave. (c) $\theta = -0.2$; wave moves to the right.

Another question is the effect of the asymmetry parameter θ. Figure 3 shows three different cases for $\theta = 0.2, 0, -0.2$. It is clearly visible that the wave speed is directly affected. Within the time $t \in [0, T]$, the wave in Figure 2(b) barely moves while there is a drift to the right in Figure 2(c) and to the left in Figure 2(a). Hence, we may conclude that the asymmetry parameter definitely has an effect on quantitative properties of traveling waves. Based on the relation to microscopic super-diffusion processes and previous studies for other nonlinearities [68], a quantitative change is expected.

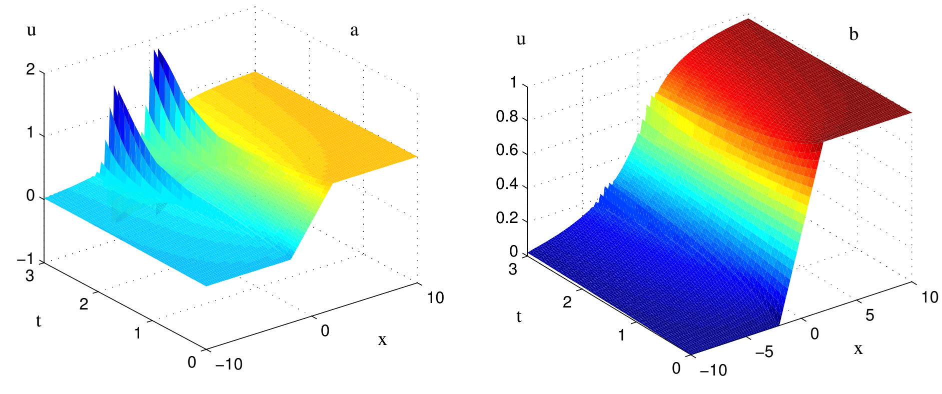

Figure 4. Fixed parameter values are $\alpha = 1.5$, $a = 0.5$, $T = 3$, $\theta = 0.4$ with $b = 10$ and $L = 101$ i.e. on a coarser grid than in the previous figures. (a) Absolute tolerance for the ODE time stepper is 10^{-6}. (b) Absolute tolerance for the ODE time stepper is 10^{-9}.

As a last issue, we briefly discuss the influence of the asymmetry parameter on numerical stability. Figure 4 shows simulations for the same parameter values $\alpha = 1.5$, $\theta = 0.4$ where θ is chosen closer to the critical line $2 - \alpha$ (see Section 2) than before. The absolute error tolerance for the numerical time step is different in Figures 4(a)-(b). Whereas we observe numerically induced oscillations in Figure 4(a) for a relatively low tolerance, the oscillations are suppressed for the more accurate computation in Figure 4(b). We checked that the numerical solution poses no problem for the lower error tolerance when θ is lower as well, for example, $\theta = 0.1$. This gives a strong indication that the ODE problem may be stiff, respectively that the region of A-stability shrinks when θ is changed. In particular, this leads to the conjecture that the asymmetric case is not only more complicated with respect to the design and implementation of numerical algorithms but also with respect to numerical stability.

3.7. *Numerical analysis: some challenges.*

In this section, we would like to highlight some numerical challenges/conjectures which are relevant for future work:

1. Provide a generalization of our scheme to higher-order quadrature rules and non-uniform meshes, including convergence and error analysis.
2. Generalize the scheme to 2- and 3-dimensional cases. What about the computation of coherent/localized structures for this case?
3. Investigate the numerical stability properties of algorithms for asymmetric fractional evolution equations regarding the (α, θ)-dependence.
4. Compare various approaches to truncate the domain $\mathbb{R}$. What is the influence of boundary conditions for space-fractional equations?
5. What about adaptive algorithms to resolve wave profiles? What is the influence of α and θ on the adaptive mesh selection?
6. Provide robust methods, including error estimates, to calculate the wave-speed and far-field/tail behavior.
7. Which methods for fractional diffusion equations, derived by different approaches such as FDM, FEM or quadrature, are equivalent?

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