

Spectral and Analytic Properties of Non-Local Schrödinger Operators and Related Jump Processes

József Lőrinczi¹, Kamil Kaleta², Samuel O. Durugo¹

¹*School of Mathematics, Loughborough University
Loughborough LE11 3TU, United Kingdom
J.Lorinczi@lboro.ac.uk, S.O.Durugo@lboro.ac.uk*

²*Institute of Mathematics, University of Warsaw,
ul. Krakowskie Przedmieście 26/28, 00-927 Warsaw
and
Institute of Mathematics and Computer Science,
Wrocław University of Technology,
Wyb. Wyspiańskiego 27, 50-370 Wrocław, Poland
kkaleta@mimuw.edu.pl, kamil.kaleta@pwr.wroc.pl*

Communicated by Gianni Pagnini and Enrico Scalas

Abstract

We discuss recent developments in the spectral theory of non-local Schrödinger operators via a Feynman-Kac-type approach. The processes we consider are subordinate Brownian motion and a class of jump Lévy processes under a Kato-class potential. We discuss some explicitly soluble specific cases, and address the spatial decay properties of eigenfunctions and the number of negative eigenvalues in the general framework of the processes we introduce.

Keywords: Lévy processes, non-local Schrödinger operators, Feynman-Kac semigroups, spectral properties, decay of eigenfunctions, Lieb-Thirring inequality

AMS subject classification: 47D08, 60G51, 47D03, 47G20

1. Introduction

1.1. Motivations and applications

Non-local operators and related random processes with jump discontinuities provide a new paradigm of scientific modelling giving a corrective to descriptions based on the usual Laplace and related second order elliptic operators. The relativistic models of quantum mechanics and field theory, sub-diffusions observed in molecular physics, anomalous transport in cell

biology, heavy-tailed and spiky distributions on the financial markets, or long-range dynamics from nano-scale to geophysical and environmental systems are all beneficiaries of these ideas and techniques. The subject is also interesting for the pure mathematician as it lies at the interface of at least three vast fields, functional analysis, stochastic analysis and partial differential equations.

An early source of these problems is relativistic quantum mechanics. Einstein's formula says that a particle moving at a high speed v whose motion is subject to the laws of classical mechanics, has kinetic energy equal to $\sqrt{m^2c^4 + m^2v^2c^2} - mc^2$, where m is the rest mass of the particle and c is the speed of light. On the other hand, quantum mechanics teaches that in order to construct the Hamilton operator describing the motion of an atomic or sub-atomic particle, the momentum $p = mv$ must be replaced by the operator $-i\nabla$ yielding the operator $\sqrt{-c^2\Delta + m^2c^4} - mc^2$, where Δ is the Laplacian. In the limit when the ratio v/c is small, the operator $-\frac{1}{2m}\Delta$ is a good approximation of the relativistic Hamiltonian, which corresponds to the quantization of the kinetic energy $\frac{mv^2}{2}$ given by Newtonian mechanics.

When one is interested in the motion of a particle which is not any longer free, extra terms appear in the Hamiltonian. For a single relativistic quantum particle under a scalar potential V , vector potential a (describing a magnetic field), and spin σ the expression becomes

$$H = (\sigma \cdot (-i\nabla - a)^2 + m^2)^{1/2} - m + V.$$

In other situations of interest one can study an electrically charged particle further coupled to a scalar or vector quantum field. When this field is a massless boson field, the model describes the interaction between the particle with a scalar radiation field such as light, and the Hamiltonian is formally written (in physical units in which $c \equiv 1$) as

$$H = \left((-\Delta + m^2)^{1/2} - m \right) \otimes 1 + 1 \otimes \int_{\mathbb{R}^d} \omega(k) a^*(k) a(k) dk \\ + \frac{1}{\sqrt{2}} \int_{\mathbb{R}^d} \frac{1}{\sqrt{\omega(k)}} \left(\hat{q}(k) e^{ik \cdot x} \otimes a(k) + \hat{q}(-k) e^{-ik \cdot x} \otimes^* a(k) \right) dk$$

where $a(k)$, $a^*(k)$ are (the formal) boson annihilation and creation operators, \hat{q} is the Fourier transform of the charge distribution of the particle, and ω is the dispersion relation.

Since the admissible energies of the particle or particle-field system are given by the eigenvalues of the Hamiltonian and the quantum states by its eigenfunctions, a fundamental problem is to study the spectrum of H . Our approach to studying spectral problems is via functional integration. In

this approach a first step is to establish a Feynman-Kac-type representation of the operator semigroup which allows to express functionals of interest by running a random process and averaging over its paths. These ideas developed to studying models of quantum field theory have been presented in the recent monograph [14], and for other models see [4,7,9].

1.2. *Some questions*

Since an explicit solution of the spectral problem is possible only for exceptional choices of operators, in our approach of studying non-local operators we are interested in the following topics:

1. *derive spectral properties*: asymptotic properties of spectrum, distribution of eigenvalues, spectral gap estimates, trace asymptotics
2. *derive analytic properties*: intrinsic ultracontractivity (IUC) of operator semigroup, heat kernel estimates, decay of eigenfunctions
3. *define and analyze Gibbs measures through the right hand side of the FK-formula*: existence, uniqueness, support properties; use Gibbs measures to establish existence and analyze ground states.

In what follows we will address some aspects of these problems. In Section 2 of this paper we introduce the two classes of Lévy processes which we will consider, the classes of potentials, and discuss Feynman-Kac-type formulae relating the given operators and processes. In Section 3 we discuss two specific cases which are explicitly soluble. In Section 4 we present ground state and other eigenfunction estimates, and discuss ground state decay on examples. In Section 5 we discuss Lieb-Thirring bounds for Bernstein functions of the Laplacian and show some explicit bounds under some reasonable conditions and present several examples.

In lack of space here we are unable to discuss literature in any detail, therefore we advise the reader to consult the references in our works.

2. Feynman-Kac semigroups for two classes of jump processes

2.1. *Bernstein functions of the Laplacian and subordinate Brownian motion*

Consider the function space

$$\mathcal{B} = \left\{ \Psi \in C^\infty(\mathbb{R}^+) : \Psi(x) \geq 0, (-1)^n \left(\frac{d^n \Psi}{dx^n} \right) (x) \leq 0, x \in \mathbb{R}^+, n \in \mathbb{N} \right\}$$

and the subclass $\mathcal{B}_0 = \{\Psi \in \mathcal{B} : \lim_{x \downarrow 0} \Psi(x) = 0\}$. An element of \mathcal{B} is a *Bernstein function*. The elements of \mathcal{B}_0 have the following integral representation. Let \mathcal{L} be the set of Borel measures λ on $\mathbb{R} \setminus \{0\}$ with the properties

that $\lambda((-\infty, 0)) = 0$ and $\int_{\mathbb{R} \setminus \{0\}} (y \wedge 1) \lambda(dy) < \infty$. Note that every $\lambda \in \mathcal{L}$ is a Lévy measure. It is known that the map $\mathcal{B}_0 \rightarrow [0, \infty) \times \mathcal{L}$, $\Psi \mapsto (b, \lambda)$ is bijective, and for every Bernstein function $\Psi \in \mathcal{B}_0$ there exists a pair $(b, \lambda) \in [0, \infty) \times \mathcal{L}$ such that

$$(2.1) \quad \Psi(x) = bx + \int_0^\infty (1 - e^{-xy}) \lambda(dy),$$

while, conversely, the right hand side of (2.1) is in \mathcal{B}_0 for each pair $(b, \lambda) \in [0, \infty) \times \mathcal{L}$.

Next consider a probability space $(\Omega_P, \mathcal{F}_P, P)$ and a subordinator $(S_t)_{t \geq 0}$ on it, i.e., $(S_t)_{t \geq 0}$ is a Lévy process starting at 0 and $t \mapsto S_t$ is almost surely a non-decreasing function. Denote by \mathcal{S} the set of subordinators on $(\Omega_P, \mathcal{F}_P, P)$ and take $\Psi \in \mathcal{B}_0$. The bijection between Bernstein functions and pairs (b, λ) given above then translates into the fact that there is a unique $(S_t)_{t \geq 0} \in \mathcal{S}$ such that

$$(2.2) \quad \mathbb{E}_P^0[e^{-uS_t}] = e^{-t\Psi(u)},$$

and, conversely, for every $(S_t)_{t \geq 0} \in \mathcal{S}$ there exists a unique $\Psi \in \mathcal{B}_0$ such that (2.2) is satisfied. In particular, (2.1) coincides with the Lévy-Khintchine formula for Laplace exponents of subordinators. Using the bijection between \mathcal{B}_0 and \mathcal{S} , we denote by $(S_t^\Psi)_{t \geq 0}$ the subordinator uniquely associated with $\Psi \in \mathcal{B}_0$.

Let $\Omega_{\mathbb{P}}$ be the canonical path space for Brownian motion $(B_t)_{t \geq 0}$ endowed with Wiener measure \mathbb{P} . The random process $X_t : \Omega_{\mathbb{P}} \times \Omega_P \ni (\omega_1, \omega_2) \mapsto B_{S_t^\Psi(\omega_2)}(\omega_1) \in \mathbb{R}^d$, $t \geq 0$, is called *d-dimensional subordinate Brownian motion* with respect to the subordinator $(S_t^\Psi)_{t \geq 0}$. It is known that every subordinate Brownian motion is a Lévy process and its properties can be determined via the formula $\mathbb{E}_{\mathbb{P} \times P}^{0,0}[e^{i\xi \cdot X_t}] = e^{-t\Psi(|\xi|^2/2)}$. We denote the density of the distribution of X_t with respect to Lebesgue measure by p_t^Ψ .

Next we consider the space of potentials which we will use. Recall that a potential $V : \mathbb{R}^d \rightarrow \mathbb{R}$ is a Borel function, acting as a multiplication operator on $L^2(\mathbb{R}^d)$. We consider it split into positive and negative parts $V = V_+ - V_-$, where $V_+ = \max\{V, 0\}$, $V_- = \min\{-V, 0\}$.

Definition 2.1 (Ψ -Kato class). *Let $\Psi \in \mathcal{B}_0$. We say that V is a Ψ -Kato class potential if $V_- \in \mathcal{K}^\Psi$ and $V_+ \mathbb{1}_B \in \mathcal{K}^\Psi$ for every ball $B \subset \mathbb{R}^d$, where we write $f \in \mathcal{K}^\Psi$ whenever*

$$(2.3) \quad \limsup_{t \downarrow 0} \sup_{x \in \mathbb{R}^d} \int_0^t \mathbb{E}_{\mathbb{P} \times P}^{x,0}[|f(B_{S_s^\Psi})|] ds = 0.$$

Remark 2.1. We note that with some extra work the above condition can be made more explicit. For instance, for the rotationally symmetric α -stable and relativistic α -stable processes condition (2.3) is further equivalent to either of the two conditions

- (1) $\lim_{\lambda \rightarrow \infty} \sup_{x \in \mathbb{R}^d} ((\Psi(-\frac{1}{2}\Delta) + \lambda)^{-1} f)(x) = 0$
- (2) $\lim_{\varepsilon \downarrow 0} \sup_{x \in \mathbb{R}^d} \int_{|x-y| < \varepsilon} R_1^\Psi(x-y) |f(y)| dy = 0,$

where by a calculation we obtain that the resolvent $R_\lambda^\Psi(x) = \int_0^\infty e^{-\lambda t} p_t^\Psi(x) dt$ is given by

$$R_\lambda^\Psi(x) = \frac{1}{(2\pi)^{d/2} |x|^{(d-1)/2}} \int_0^\infty \frac{r^{(d-1)/2}}{\lambda + \Psi(r^2/2)} \sqrt{r|x|} J_{(d-2)/2}(r|x|) dr,$$

with the Bessel function of the first kind $J_\nu(z)$. For further expressions and other processes see [5].

Let $\Psi \in \mathcal{B}_0$, and V be a potential such that V_- is form-bounded with respect to $\Psi(-\frac{1}{2}\Delta)$ with a relative bound strictly smaller than 1, and $V_+ \in L_{\text{loc}}^1(\mathbb{R}^d)$. Then we define the *Schrödinger operator with Bernstein function Ψ of the Laplacian* by the quadratic form sum

$$(2.4) \quad H^\Psi = \Psi(-\frac{1}{2}\Delta) \dot{+} V_+ \dot{-} V_-.$$

Example 2.1. In view of applications (quantum theory, anomalous transport theory, financial mathematics etc) some particular choices of Bernstein functions are of special interest involving the following stochastic processes:

- (1) *standard Brownian motion:* $\Psi(u) = 2u$
- (2) *rotationally symmetric non-Gaussian α -stable processes:* $\Psi(u) = (2u)^{\alpha/2}, 0 < \alpha < 2$
- (3) *sums of independent rotationally symmetric stable processes of different index:* $\Psi(u) = a(2u)^{\alpha/2} + b(2u)^{\beta/2}, 0 < \alpha < \beta < 2, a, b > 0$
- (4) *jump-diffusion processes:* $\Psi(u) = au + bu^{\alpha/2}$, with $0 < \alpha < 2, a, b > 0$
- (5) *rotationally symmetric relativistic α -stable processes:* $\Psi(u) = (2u + m^{2/\alpha})^{\alpha/2} - m$, with $0 < \alpha < 2, m > 0$
- (6) *rotationally symmetric geometric α -stable processes:* $\Psi(u) = \log(1 + u^{\alpha/2}), 0 < \alpha \leq 2.$

For a detailed discussion and a catalogue of 138 Bernstein functions we refer to [16]. In particular, the Feynman-Kac representation of the evolution semigroup generated by a *fractional Schrödinger operator*

$$(2.5) \quad H = (-\Delta)^{\alpha/2} + V$$

involves the process (1) perturbed by the potential V , and a *relativistic fractional Schrödinger operator*

$$(2.6) \quad H = (-\Delta + m^{2/\alpha})^{\alpha/2} - m + V, \quad m > 0$$

involves (4). For $\alpha = 1$ these operators describe a massless resp. massive relativistic particle in a potential V , and the related processes are Cauchy resp. relativistic Cauchy processes perturbed by V .

We have the following relationships between the evolution semigroup generated by H^Ψ and a random process. For details of proof we refer to [5,14].

Theorem 2.1 (Feynman-Kac-type representation: bounded potential).

Let $\Psi \in \mathcal{B}_0$ and consider the subordinator $(S_t^\Psi)_{t \geq 0}$. If $V \in L^\infty(\mathbb{R}^d)$, then the semigroup $\{e^{-tH^\Psi} : t \geq 0\}$ has the integral representation

$$(2.7) \quad (f, e^{-tH^\Psi} g) = \int_{\mathbb{R}^d} dx \mathbb{E}_{\mathbb{P} \times P}^{x,0} \left[e^{-\int_0^t V(B_{S_s^\Psi}) ds} f(B_0) g(B_{S_t^\Psi}) \right], \quad f, g \in L^2(\mathbb{R}^d).$$

This formula can be extended to Ψ -Kato class potentials.

Theorem 2.2 (Feynman-Kac-type representation: Kato-class potential).

Let $V \in \mathcal{B}_0$ and consider the subordinator $(S_t^\Psi)_{t \geq 0}$. For a Ψ -Kato class potential V define

$$(U_t^\Psi f)(x) = \mathbb{E}_{\mathbb{P} \times P}^{x,0} \left[e^{-\int_0^t V(B_{S_s^\Psi}) ds} f(B_{S_t^\Psi}) \right], \quad f \in L^2(\mathbb{R}^d).$$

Then

- (1) $\{U_t^\Psi : t \geq 0\}$ is a strongly continuous symmetric semigroup
- (2) there exists a self-adjoint operator K^Ψ , bounded from below, such that $U_t^\Psi = e^{-tK^\Psi}$.

The above result allows to view K^Ψ as a *non-local Schrödinger operator with a Bernstein function of the Laplacian and Ψ -Kato potential*. We can furthermore prove that under some reasonable conditions on Ψ and p_t^Ψ the operator K^Ψ coincides with the operator given by (2.4).

In [5] we have obtained further Feynman-Kac-type representations for Schrödinger operators with Bernstein functions of the Laplacian including interactions with a magnetic field and spin. Also, we refer to this paper for a discussion of $L^p - L^q$ bounds of the Feynman-Kac semigroup, diamagnetic inequalities, a new type of energy comparison inequality etc.

2.2. A class of Lévy processes with controlled large jumps

A second class of Lévy processes will be given by conditions in terms of their Lévy measure and transition probability densities rather than a generator.

Let $D(\mathbb{R}^+, \mathbb{R}^d)$ be the space of real-valued càdlàg paths over the positive semi-axis and consider a Lévy process $(X_t)_{t \geq 0}$ on this space. We assume that the Lévy measure of $(X_t)_{t \geq 0}$ is $\nu(dx) = \nu(x)dx$, i.e., absolutely continuous with respect to Lebesgue measure, with density $\nu(x) > 0$, for all $x \in \mathbb{R}^d$. The class of processes we consider has been introduced in [12].

Assumption 2.1.

(1) *The following assumptions hold on the Lévy intensity:*

(i) *for every $0 < r \leq 1/2$ there is a constant $C_1 = C_1(X, r) \geq 1$ such that*

$$\nu(x) \asymp C_1 \nu(y), \quad r \leq |y| \leq |x| \leq |y| + 1$$

(ii) *there is a constant $C_2 = C_2(X) \geq 1$ such that*

$$\nu(x) \leq C_2 \nu(y), \quad 1/2 \leq |y| \leq |x|.$$

(iii) *there is a constant $C_3 = C_3(X) \geq 1$ such that*

$$\int_{\substack{|z-x| > 1/2 \\ |z-y| > 1/2}} \nu(x-z)\nu(z-y)dz \leq C_3 \nu(x-y), \quad |y-x| \geq 1.$$

(2) *There exist the transition probability densities $p(t, x, y) =: p(t, y-x)$, i.e., for every $x \in \mathbb{R}^d$, $t > 0$ and Borel set $A \subset \mathbb{R}^d$ we have $P^x(X_t \in A) = \int_A p(t, x, y)dy$, where P^x is the measure of $(X_t)_{t \geq 0}$ starting at x . Furthermore, we assume that there exist $t_b > 0$ and $C_4 = C_4(X, t_b)$ such that $0 < p(t_b, x) \leq C_4$, for all $x \in \mathbb{R}^d$.*

(3) *For all $0 < p < q < R \leq 1$ we have $\sup_{x \in B(0,p)} \sup_{y \in B_q(0)^c} G_{B_R(0)}(x, y) < \infty$, where $G_{B_R(0)}(x, y) = \int_0^\infty p_{B_R(0)}(t, x, y)dt$ denotes the Green function of the process $(X_t)_{t \geq 0}$ in the ball $B_R(0)$.*

Example 2.2. For a detailed discussion of these conditions and numerous examples we refer to [12]. The Lévy processes covered by Assumption 2.1 have a non-trivial overlap with the class of subordinate Brownian motions, however, neither class contains the other. Here we give a few examples and counterexamples only.

(1) The specific cases (2)-(5) in Example 2.1 and (6) with $0 < \alpha < 2$ all satisfy Assumption 2.1.

- (2) Processes with exponentially and subexponentially localized Lévy measure $\nu(x) \asymp e^{-a|x|^\beta} |x|^{-d-\delta} (1+|x|)^{d+\delta-\gamma}$ with $a > 0$ satisfy Assumption 2.1 whenever $\beta \in (0, 1]$, $\delta \in (0, 2)$ and $\gamma > (d+1)/2$.
- (3) For $\beta > 1$ in case example (2) above Assumption 2.1 is not satisfied. It fails to hold also in the case of the rotationally symmetric 2-geometric stable process, i.e., when $\beta = 1$, $\gamma = (d+1)/2$ and $\delta = 0$.

Note that it has been recently established in [13] that the convolution condition (iii) in Assumption 2.1 (1) characterizes the large class of Lévy processes for which $p_t(x) \asymp t\nu(x)$ for small t and large x .

The potentials we consider in the case of Lévy processes with controlled large jumps will be required to satisfy the following conditions.

Assumption 2.2.

- (1) V is an X -Kato class potential, i.e., $V_- \in \mathcal{K}^X$ and $V_+ \mathbb{1}_B \in \mathcal{K}^X$ for every ball $B \subset \mathbb{R}^d$, where we write $f \in \mathcal{K}^X$ whenever $\lim_{t \downarrow 0} \sup_{x \in \mathbb{R}^d} \mathbb{E}^x \left[\int_0^t |f(X_s)| ds \right] = 0$, and where the expectation is taken with respect to the measure of $(X_t)_{t \geq 0}$.
- (2) V is a confining potential, i.e., $V(x) \rightarrow \infty$ as $|x| \rightarrow \infty$.

By an extension of Khasminskii’s lemma (compare [14, Lemma 3.37]) to X -Kato potentials it follows that the random variables $\int_0^t V(X_s) ds$ are exponentially integrable for all $t \geq 0$, and thus we can define the Feynman-Kac semigroup

$$(2.8) \quad T_t f(x) = \mathbb{E}^x \left[e^{-\int_0^t V(X_s) ds} f(X_t) \right], \quad f \in L^2(\mathbb{R}^d), \quad t > 0.$$

Using the Markov property and stochastic continuity of the process it can be shown that $\{T_t : t \geq 0\}$ is a strongly continuous one-parameter semigroup of symmetric operators on $L^2(\mathbb{R}^d)$. Moreover, by the Hille-Yoshida theorem there exists a self-adjoint operator H bounded from below such that $e^{-tH} = T_t$, which may be viewed as another generalization to a *non-local Schrödinger operator*.

Note that due to condition (2) in Assumption 2.2 the generator H has a purely discrete spectrum. An eigenfunction $\varphi_1 \in L^2(\mathbb{R}^d)$ such that $H\varphi_1 = \lambda_1 \varphi_1$, where $\lambda_1 = \inf \text{Spec } H$ is the bottom of the spectrum of H , is called a ground state of H . By standard arguments [14, Sect. 3.4.3] it follows that φ_1 is unique and has a strictly positive version, which will be used throughout.

3. Some explicitly tractable examples

3.1. Massless relativistic harmonic oscillator

Explicit solutions of the eigenvalue problem for Schrödinger operators are known for a very few specific potentials only. It is clearly desirable to derive similar explicit formulae also for non-local Schrödinger operators. Such examples have been explored in [3,15].

Consider the operator

$$(3.1) \quad H = \sqrt{-\frac{d^2}{dx^2} + x^2}$$

on $L^2(\mathbb{R})$. Here the potential is $V(x) = x^2$, so H has a purely discrete spectrum consisting of eigenvalues $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots \rightarrow \infty$, each having finite multiplicity. The corresponding eigenfunctions $\varphi_1, \varphi_2, \dots$ are bounded continuous functions and form an orthonormal basis in $L^2(\mathbb{R})$. Also, we can show that $\varphi_n \in L^1(\mathbb{R})$, therefore by Fourier transform $\widehat{\varphi}_n(y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ixy} \varphi_n(x) dx$ the eigenvalue equation becomes

$$-\widehat{\varphi}_n''(y) + |y| \widehat{\varphi}_n(y) = \lambda_n \widehat{\varphi}_n(y).$$

An argument shows that $\widehat{\varphi}_n \in C^2(\mathbb{R}) \cap L^1(\mathbb{R})$ for all n . Using these conditions and the ODE above we are able to determine the spectrum of H and the Fourier transform of the eigenfunctions.

Theorem 3.1 (Spectrum and eigenfunctions). *The eigenvalues of H are given by*

$$\lambda_{2k-1} = -a'_k \quad \text{and} \quad \lambda_{2k} = -a_k, \quad k \in \mathbb{N},$$

where a_k and a'_k denote the zeroes of the Airy function Ai and its derivative Ai' in decreasing order. They are all simple, the eigenfunctions φ_{2k-1} are even and φ_{2k} are odd. Moreover, the Fourier transforms of the L^2 -normalized eigenfunctions are given by

$$\widehat{\varphi}_n(y) = \begin{cases} \frac{\text{Ai}(|y| - \lambda_n)}{\sqrt{2\lambda_n} \text{Ai}(-\lambda_n)} & \text{if } n = 1, 3, 5, \dots \\ \frac{\text{sgn}(y) \text{Ai}(|y| - \lambda_n)}{\sqrt{2} \text{Ai}'(-\lambda_n)} & \text{if } n = 2, 4, 6, \dots \end{cases}, \quad y \in \mathbb{R}.$$

To have a closer idea about how the eigenvalues actually depend on n , we can make use of the asymptotic expansions and estimates for the zeroes

of the Airy function and its derivative. We obtain the following expressions.

Corollary 3.1. *For $k \in \mathbb{N}$, we have the bounds*

$$\lambda_{2k-1} \leq \left(\frac{3\pi}{8}(4k-1) \right)^{2/3}$$

$$\left(\frac{3\pi}{8}(4k-1) \right)^{2/3} \leq \lambda_{2k} \leq \left(\frac{3\pi}{8}(4k-1) \right)^{2/3} \left(1 + \frac{3}{2} \arctan \left(\frac{5}{18\pi(4k-1)} \right) \right).$$

Furthermore, in the limit as $k \rightarrow \infty$ we have asymptotically

$$\lambda_{2k-1} \sim g \left(\frac{3\pi}{8}(4k-1) \right) \quad \text{and} \quad \lambda_{2k} \sim f \left(\frac{3\pi}{8}(4k-1) \right),$$

where

$$g(u) = u^{2/3} \left(1 - \frac{7}{48}u^{-2} + \frac{35}{288}u^{-4} - \frac{181223}{207360}u^{-6} + \frac{18683371}{1244160}u^{-8} - \dots \right),$$

$$f(u) = u^{2/3} \left(1 + \frac{5}{48}u^{-2} - \frac{5}{36}u^{-4} + \frac{77125}{82944}u^{-6} - \frac{108056875}{6967296}u^{-8} + \dots \right).$$

Corollary 3.2. *From the above eigenvalue estimates we furthermore obtain*

- (1) spectral gap estimate: $\lambda_2 - \lambda_1 \geq \left(\frac{3\pi}{8} \right)^{2/3} (3^{2/3} - 1)$
- (2) heat trace asymptotics: $\lim_{t \rightarrow 0^+} t^{3/2} \sum_{n \geq 1} e^{-\lambda_n t} = \frac{1}{\sqrt{\pi}}$.

The next result gives a full asymptotic expansion of the eigenfunctions.

Theorem 3.2. *For every $k \in \mathbb{N}$ and $N = 2, 3, \dots$ we have as $|x| \rightarrow \infty$ that*

$$\varphi_{2k-1}(x) = \sqrt{\frac{2}{-a'_k}} \sum_{j=2}^N (-1)^j \frac{p_{2j-1}(a'_k)}{x^{2j}} + O\left(\frac{1}{x^{2N+2}}\right)$$

$$\varphi_{2k}(x) = \sqrt{2} \sum_{j=2}^N (-1)^j \frac{q_{2j}(a_k)}{x^{2j+1}} + O\left(\frac{1}{x^{2N+3}}\right)$$

where p_n, q_n are n th order polynomials defined by the recursive relations $p_{n+1}(x) = p'_n(x) + xq_n(x)$ and $q_{n+1}(x) = p_n(x) + q'_n(x)$, with $p_0(x) \equiv 1$, $q_0(x) \equiv 0$.

We conclude by summarizing some further properties of the eigenfunctions, for details and other results we refer to [15].

Theorem 3.3. For all $n \in \mathbb{N}$

- (1) φ_n is an analytic function (Maclaurin expansions obtained)
- (2) φ_n has a finite number of zeroes
- (3) φ_n is uniformly bounded.

Moreover, the ground state φ_1 is decreasing on $(0, \infty)$, concave on $[-x_0, x_0]$, and convex on $(-\infty, -x_1] \cup [x_1, \infty)$, where x_0 and x_1 are suitable numbers.

3.2. Massless relativistic quartic anharmonic oscillator

Another specific case that can be treated explicitly is the model of the quartic oscillator $V(x) = x^4$ for which the Hamiltonian is given by

$$(3.2) \quad H = \sqrt{-\frac{d^2}{dx^2}} + x^4.$$

This too has purely eigenvalues and by a similar method as before the eigenvalue problem transforms into the equation

$$(3.3) \quad \frac{d^4}{dy^4} \widehat{\varphi}_n(y) + |y| \widehat{\varphi}_n(y) = \lambda_n \widehat{\varphi}_n(y),$$

now with $\widehat{\varphi}_n \in C^4(\mathbb{R}) \cap L^1(\mathbb{R})$ for all n . The boundary conditions imply that the symmetric and antisymmetric solutions must satisfy

$$(3.4) \quad \widehat{\varphi}'_{2k-1}(0^+) = \widehat{\varphi}'''_{2k-1}(0^+) = 0 \quad \text{and} \quad \widehat{\varphi}_{2k}(0^+) = \widehat{\varphi}''_{2k}(0^+) = 0.$$

Unlike in the harmonic case, the solution of (3.3) is obtained in expressions containing two terms. The first is the so called fourth order extended Airy function of the first kind

$$(3.5) \quad \text{Ai}_4(y) = \frac{1}{\pi} \int_0^\infty \cos\left(\frac{t^5}{5} + yt\right) dt,$$

and the other term is

$$(3.6) \quad \widetilde{\text{Ai}}_4(y) = \frac{1}{\pi} \int_0^\infty \left[e^{-yt - \frac{t^5}{5}} - \sin\left(\frac{t^5}{5} + yt\right) \right] dt$$

obtained by combinations of $\text{Ai}_4(y)$ with rotated argument.

The function Ai_4 appears to be little explored in the literature and has been analyzed in detail in [1,2]. Figure 1 plots $\text{Ai}_4(y)$ and $\widetilde{\text{Ai}}_4(y)$ below. Figure 1 (left) shows the extended Airy function decaying faster than an exponential function on $[0, \infty)$ with damped oscillation on $(-\infty, 0)$. Like the classical Airy function $\text{Ai}(y)$, this function has an infinite number of

negative zeroes but also has a real positive zero whose value is up to a small error equal to the first real negative zero. With $\widetilde{\text{Ai}}_4(y)$ as shown in Figure 1 (right), we obtain a function decaying faster than an exponential function on $[0, \infty)$ and also decaying faster than an exponential function on $(-\infty, 0)$. The plot shows the dominance of the integral involving the exponential term in (3.6) on both half axes of the real line.

Using these functions and (3.4) we obtain that the spectrum of the quartic oscillator are given by the zeroes of the functions

$$\Phi_1(y) = \det \begin{pmatrix} \text{Ai}_4(y) & \widetilde{\text{Ai}}_4(y) \\ \text{Ai}_4''(y) & \widetilde{\text{Ai}}_4''(y) \end{pmatrix} \quad \text{and} \quad \Phi_2(y) = \det \begin{pmatrix} \text{Ai}_4'(y) & \widetilde{\text{Ai}}_4'(y) \\ \text{Ai}_4'''(y) & \widetilde{\text{Ai}}_4'''(y) \end{pmatrix}$$

which can be further expressed in terms of the generalized Fresnel sine and cosine integrals. These findings can be summarized as follows, for details see [3].

Theorem 3.4 (Spectrum and eigenfunctions). *The eigenvalues of the quartic oscillator H are given by*

$$\lambda_{2k-1} = -\alpha_{2,k} \quad \text{and} \quad \lambda_{2k} = -\alpha_{1,k}, \quad k \in \mathbb{N},$$

where $\alpha_{1,k}$ and $\alpha_{2,k}$ denote the negative real zeroes of the determinant functions $\Phi_1(y)$ and $\Phi_2(y)$, respectively, which are simple and arranged in increasing order. The eigenfunctions φ_{2k-1} are even and φ_{2k} are odd, and their Fourier transforms are given by

$$\widehat{\varphi}_n(y) = \begin{cases} c_{1,n} \text{Ai}_4(|y| - \lambda_n) + c_{2,n} \widetilde{\text{Ai}}_4(|y| - \lambda_n), & n = 2k - 1 \\ c_{1,n} \text{sgn}(y) \text{Ai}_4(|y| - \lambda_n) + c_{2,n} \text{sgn}(y) \widetilde{\text{Ai}}_4(|y| - \lambda_n), & n = 2k \end{cases}$$

with $k \in \mathbb{N}$, and normalization constants

$$(3.7) \quad c_{1,n} := \begin{cases} -\frac{1}{\sqrt{2}} \frac{\widetilde{\text{Ai}}_4'(-\lambda_n)}{\sqrt{\lambda_n \Lambda_1^2(-\lambda_n) + \Lambda_2^2(-\lambda_n)}} & n = 1, 3, 5, \dots \\ \frac{1}{2} \frac{\widetilde{\text{Ai}}_4''(-\lambda_n)}{\sqrt{\Lambda_2(-\lambda_n) \Lambda_3(-\lambda_n)}} & n = 2, 4, 6, \dots \end{cases}$$

and

$$(3.8) \quad c_{2,n} := \begin{cases} \frac{1}{\sqrt{2}} \frac{\text{Ai}_4'(-\lambda_n)}{\sqrt{\lambda_n \Lambda_1^2(-\lambda_n) + \Lambda_2^2(-\lambda_n)}} & n = 1, 3, 5, \dots \\ -\frac{1}{2} \frac{\text{Ai}_4''(-\lambda_n)}{\sqrt{\Lambda_2(-\lambda_n) \Lambda_3(-\lambda_n)}} & n = 2, 4, 6, \dots \end{cases}$$

where $\Lambda_1(\xi) := \widetilde{\text{Ai}}_4(\xi) \text{Ai}_4'(\xi) - \text{Ai}_4(\xi) \widetilde{\text{Ai}}_4'(\xi)$, $\Lambda_2(\xi) := \widetilde{\text{Ai}}_4'(\xi) \text{Ai}_4''(\xi) - \text{Ai}_4'(\xi) \widetilde{\text{Ai}}_4''(\xi)$, $\Lambda_3(\xi) := \widetilde{\text{Ai}}_4''(\xi) \text{Ai}_4'''(\xi) - \text{Ai}_4''(\xi) \widetilde{\text{Ai}}_4'''(\xi)$.

From here we obtain

Theorem 3.5. *We have*

$$\lambda_n = \left(\frac{5(2n-1)\pi}{16} \right)^{4/5} \left[1 + O\left(\frac{1}{n^{3/2}} \right) \right] \quad \text{as } n \rightarrow \infty.$$

It is remarkable that due to the small effect from $\widetilde{\text{Ai}}_4$, the true spectrum is only slightly shifted from the zeroes of Ai_4 . In the table below we illustrate this by some numerical values. Like in the case of the quadratic potential, further results on the spectrum and eigenfunctions can be obtained, which will be discussed elsewhere.

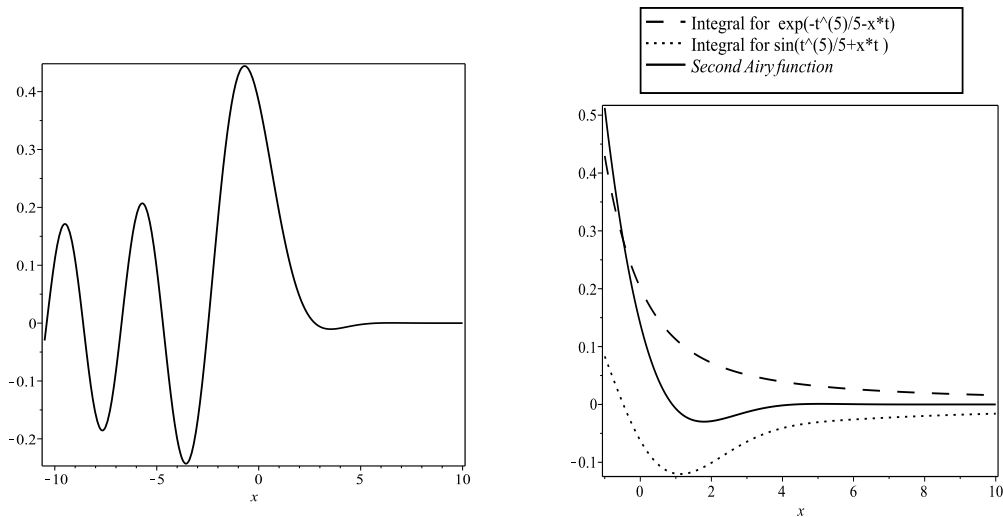


Figure 1: Plots of $\text{Ai}_4(x)$ (left) and $\widetilde{\text{Ai}}_4(x)$ (right)

4. Eigenfunction decay

For a ground state transformed process related to $(X_t)_{t \geq 0}$ the square of the ground state gives the density of the stationary distribution (see e.g. [11]). The mass concentration properties of this density translate in quantum theory applications into information about the localization of the particle in space. Therefore the decay of ground state is of special interest which we discuss next.

Due to its interest in quantum theory, this problem has been first addressed in the literature for relativistic Schrödinger operators (2.5)-(2.6)

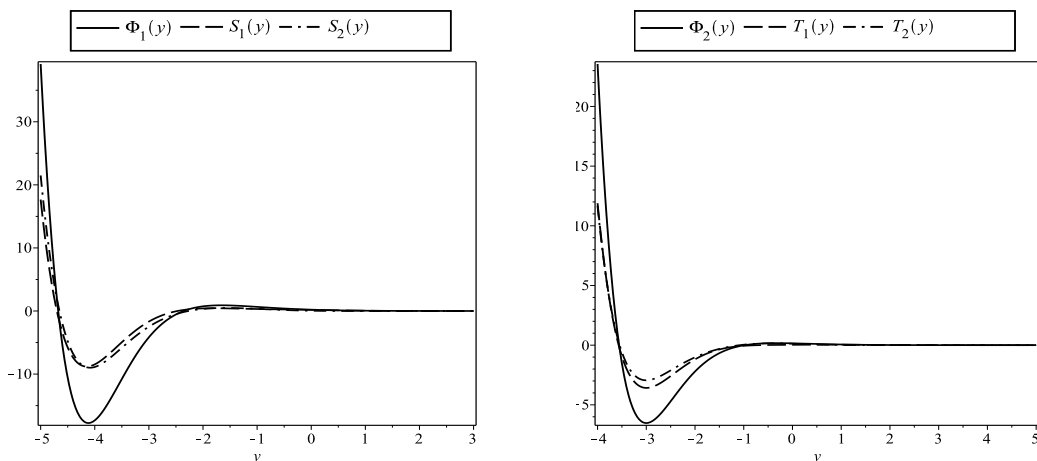


Figure 2: Determinant functions $\Phi_1(y)$ (left) and $\Phi_2(y)$ (right) and components

Table 1: Numerical computations of eigenvalues λ_n for the quadratic and quartic cases and the zeroes of Ai_4 and Ai'_4 .

n	$\lambda_n^{(2)}$	$\lambda_n^{(4)}$	Zeroes of Ai_4 (<i>slanted</i>) and Ai'_4
1	1.01879297	0.97842695	0.69054021
2	2.33810741	2.35819511	<i>2.47328205</i>
3	3.24819758	3.55762902	3.56708317
4	4.08794944	4.67603019	<i>4.70001675</i>
5	4.82009921	5.70790501	5.70805278
6	5.52055983	6.71126508	<i>6.72601722</i>
7	6.16330736	7.66488539	7.66503104
8	6.78670809	8.60054100	<i>8.61061536</i>
9	7.37217726	9.50196086	9.50201767
10	7.94413359	10.39051635	<i>10.39807977</i>
—	—	—	—
15	10.52766040	14.57028767	14.57029807
25	14.93593720	22.16864714	22.16864908
50	23.87156446	38.91432285	<i>38.91535535</i>
60	26.37880505	45.08566312	<i>45.08649070</i>

with $\alpha = 1$, and later for all $0 < \alpha < 2$ (see [10,11] and the references therein). Models also including (possibly unbounded) magnetic field and spin have been discussed in [6] for both confining and decaying potentials by developing martingale methods. The ground state decay for the largest class of processes to date has been analyzed in [12] by using methods of

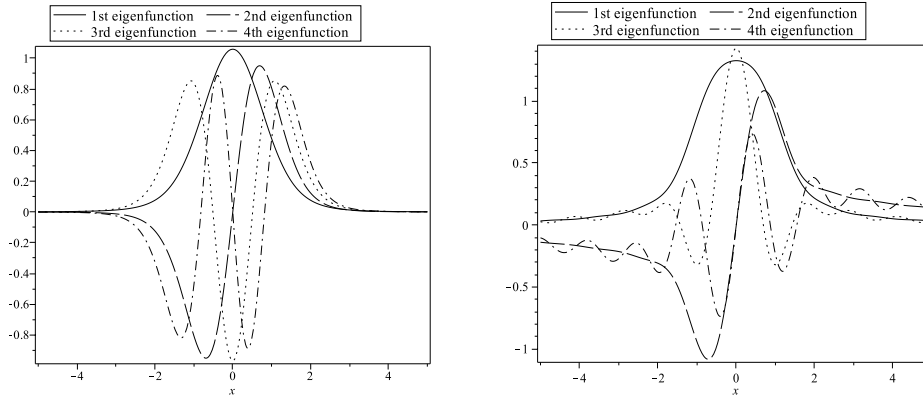


Figure 3: Plots of first four eigenfunctions for the harmonic and the quartic anharmonic oscillators cases

potential theory. We have obtained pointwise upper bounds for eigenfunctions and sharp two-sided bounds for the ground state of the operators T_t given by (2.8).

Let $D \subset \mathbb{R}^d$ be an open set, V a Kato-class potential for a given process $(X_t)_{t \geq 0}$, and φ a non-negative or bounded Borel function. Recall that the V -Green operator for the semigroup $\{T_t : t \geq 0\}$ and set D is defined by

$$G_D^V \varphi(x) = \mathbb{E}^x \left[\int_0^{\tau_D} e^{-\int_0^t V(X_s) ds} \varphi(X_t) dt \right], \quad x \in D,$$

where $\tau_D = \inf \{t \geq 0 : X_t \notin D\}$ is the first exit time from D . Then we have the following results (with the notations introduced in Section 2).

Theorem 4.1 (Upper bounds on eigenfunctions).

If Assumptions 2.1 and 2.2 hold, then for every $n \in \mathbb{N}$ and $\eta \geq 0$ such that $\lambda_1 + \eta > 0$, there exists a constant $C(X, V, n, \eta) > 0$ and a radius $R = R(X, V, n, \eta) > 0$ such that

$$|\varphi_n(x)| \leq G_{B(x,1)}^{V+\eta} \mathbb{1}(x) \nu(x), \quad |x| \geq R.$$

For the ground state we have furthermore sharp lower and upper estimates.

Theorem 4.2 (Ground state estimates). If Assumptions 2.1 and 2.2 hold, then for every $\eta \geq 0$ such that $\lambda_1 + \eta > 0$ there exist constants $C(X, V, \eta), C'(X, V, \eta) > 0$ and a radius $R = R(X, V, \eta) > 0$ such that

$$C G_{B(x,1)}^{V+\eta} \mathbb{1}(x) \nu(x) \leq \varphi_1(x) \leq C' G_{B(x,1)}^{V+\eta} \mathbb{1}(x) \nu(x), \quad |x| \geq R.$$

The following ground state domination property is an immediate consequence of the above theorems. We note that this is in contrast with Brownian motion in a potential, for which it does not occur if the growth of V at infinity is not fast enough (an example is the harmonic oscillator with $V(x) = |x|^2$).

Corollary 4.1. *If Assumptions 2.1 and 2.2 hold, then for every $n = 2, 3, \dots$ there is a constant $C_n(X, V) > 0$ such that*

$$|\varphi_n(x)| \leq C_n \varphi_1(x), \quad x \in \mathbb{R}^d.$$

From the above estimates we also have the following corollary.

Corollary 4.2. *Let Assumptions 2.1 and 2.2 hold. Then for every $n \in \mathbb{N}$ there exists a radius $R = R(X, V, n) > 0$ and a constant $C(X, V) > 0$ such that*

$$|\varphi_n(x)| \leq C \frac{\nu(x)}{\inf_{y \in B(x, 1)} V(y)}, \quad |x| \geq R,$$

and

$$c \frac{\nu(x)}{\sup_{y \in B(x, 1)} V(y)} \leq \varphi_1(x) \leq c' \frac{\nu(x)}{\inf_{y \in B(x, 1)} V(y)}, \quad |x| \geq R,$$

with some constants $c(X, V), c'(X, V)$. In particular, if for some $n \in \mathbb{N}$ there is a constant $C > 1$ such that for all unit balls $B \subset B(0, R)^c$ it holds that $\sup_{y \in B} V(y) \leq C \inf_{y \in B} V(y)$, then

$$|\varphi_n(x)| \leq \text{const} \frac{\nu(x)}{V(x)}, \quad |x| \geq R + 1,$$

and

$$K_1 \frac{\nu(x)}{V(x)} \leq \varphi_1(x) \leq K_2 \frac{\nu(x)}{V(x)}, \quad |x| \geq R + 1,$$

with some constants $K_1(X, V), K_2(X, V) > 0$.

Example 4.1.

1. *Diffusions.* For diffusions generated by Schrödinger operators $-\frac{1}{2}\Delta + V$ there is a large literature on ground state decay. In this case the typical result is exponential decay, even for potentials going to zero at infinity. For confining potentials a representative result is the case of $V(x) \asymp |x|^{2n}$, $n \geq 1$, giving $\varphi_1(x) \asymp e^{-c|x|^{n+1}}$.

2. *Rotationally symmetric non-Gaussian stable and related processes.* This class includes cases (2)-(4) and (6) with $0 < \alpha < 2$ in Example 2.1, and others. For this family we have $\nu(x) \asymp |x|^{-d-\alpha}$ for large $|x|$, hence it follows that

$$\varphi_1(x) \asymp G_{B(x,1)}^V \mathbb{1}(x) |x|^{-d-\alpha}, \quad |x| > R,$$

where R is large enough. Furthermore, when also the condition in Corollary 4.2 is satisfied, then

$$\varphi_1(x) \asymp \frac{1}{(1 + |x|)^{d+\alpha}(1 + V_+(x))}, \quad x \in \mathbb{R}^d.$$

3. *Rotationally symmetric jump Lévy processes with exponentially localized Lévy measure.* Whenever $e^{-t\Psi} \in L^1(\mathbb{R}^d)$, $t > 0$, and there is $b > 0$ such that $\int_{|x|>1} e^{b|x|} \nu(dx) < \infty$, then $|\varphi_n(x)| \leq C e^{-C'|x|}$, $x \in \mathbb{R}^d$, with $C = C(X, V, n)$, $C' = C'(X, V, n)$, i.e., if the Lévy measure is exponentially localized, then so is the the fall-off of the ground state. In the case of the relativistic stable processes (5) in Example 2.1 we obtain

$$\varphi_1(x) \asymp \frac{e^{-m^{1/\alpha}|x|}}{(1 + |x|)^{(d+\alpha+1)/2}(1 + V_+(x))}, \quad x \in \mathbb{R}^d.$$

5. Lieb-Thirring bounds

In this section we consider Bernstein functions of the Laplacian as given in (2.4), choose $d \geq 3$, and make the following assumptions on the potential.

Assumption 5.1.

- (1) V is a continuous and non-positive function
- (2) there exists $\lambda^* > 0$ such that $\|(\Psi(-\frac{1}{2}\Delta) + \lambda)^{-1/2}|V|^{1/2}\| < 1$ for all $\lambda \geq \lambda^*$
- (3) the operator $(\Psi(-\frac{1}{2}\Delta) + \lambda)^{-1/2}|V|^{1/2}$ is compact for all $\lambda \geq 0$
- (4) there exists $n_0 > 0$ such that $\text{Tr}(|V|^{1/2}(\Psi(-\frac{1}{2}\Delta) + \lambda)^{-1}|V|^{1/2})^n < \infty$ for all $n \geq n_0$ and $\lambda > 0$.

Part (2) of Assumption 5.1 implies that V is relatively form bounded with respect to $\Psi(-\frac{1}{2}\Delta)$ with relative bound strictly smaller than 1. Part (3) ensures that the Birman-Schwinger principle (5.3) holds (see below), and (4) is a technical condition.

Consider the number

$$(5.1) \quad N_E(V) = \dim \mathbb{1}_{(-\infty, -E]}(H^\Psi),$$

counting the eigenvalues up to a negative level $-E$, which in the original context of quantum theory is the number of bound states with energy less or equal than this value. Recall that the Birman-Schwinger kernel is defined by

$$(5.2) \quad K_E = |V|^{1/2} \left(\Psi(-\frac{1}{2}\Delta) + E \right)^{-1} |V|^{1/2}$$

and the *Birman-Schwinger principle* states that

$$(5.3) \quad \begin{aligned} N_E(V) &= \dim \mathbb{1}_{[1, \infty)}(K_E), \quad -E < 0 \\ N_0(V) &\leq \dim \mathbb{1}_{[1, \infty)}(K_0), \quad E = 0. \end{aligned}$$

Let $F_\lambda(x) = x(1 + \lambda x)^{-1} = x \int_0^\infty e^{-y(1+\lambda x)} dy$ and $g_\lambda(x) = e^{-\lambda x}$, so that we have $F_\lambda(x) = x \int_0^\infty e^{-y} g_\lambda(xy) dy$. A direct computation gives

$$\begin{aligned} (F_\lambda(K_E)f)(x) &= \left(|V|^{1/2} (\Psi(-\frac{1}{2}\Delta) + \lambda|V| + E)^{-1} |V|^{1/2} f \right)(x) \\ &= |V(x)|^{1/2} \left(\int_0^\infty dt e^{-tE} e^{-t(\Psi(-\frac{1}{2}\Delta) + \lambda|V|)} |V|^{1/2} f \right)(x) \end{aligned}$$

for all $f \in L^2(\mathbb{R}^d)$. By (5.3) we have

$$\begin{aligned} N_E(V) &= \#\{F_\lambda(\mu) | F_\lambda(\mu) \text{ is an eigenvalue of } F_\lambda(K_E) \text{ and } \mu \geq 1\}, \quad E > 0 \\ N_0(V) &\leq \#\{F_\lambda(\mu) | F_\lambda(\mu) \text{ is an eigenvalue of } F_\lambda(K_0) \text{ and } \mu \geq 1\}, \quad E = 0. \end{aligned}$$

Since F_λ is monotone increasing, it follows that

$$N_E(V) \leq \frac{1}{F_\lambda(1)} \sum_{\substack{\mu \in \text{Spec}(K_E) \\ \mu \geq 1}} F_\lambda(\mu).$$

Next we will estimate the trace of $F_\lambda(K_E)$. The Feynman-Kac formula in Theorem 2.8 allows the expression

$$(5.4) \quad (F_\lambda(K_E)f)(x) = |V(x)|^{1/2} \int_0^\infty dt e^{-tE} \mathbb{E}_{\mathbb{P} \times P}^x \left[e^{-\lambda \int_0^t |V(X_s)| ds} |V(X_t)|^{1/2} f(X_t) \right].$$

Since the kernel

$$e^{-t(\Psi(-\frac{1}{2}\Delta) + \lambda|V|)}(x, y) = \mathbb{E}_{\mathbb{P} \times P}^0 \left[e^{-\lambda \int_0^t |V(X_s+x)| ds} \Big| X_t + x = y \right] p_t^\Psi(x - y),$$

where recall that p_t^Ψ is the density of the distribution of X_t , and we can show that the map $(x, y) \mapsto e^{-t(\Psi(-\frac{1}{2}\Delta)+|V|)}(x, y)$ is continuous, we obtain that that the kernel

$$F_\lambda(K_E)(x, y) = |V(x)|^{1/2}|V(y)|^{1/2} \int_0^\infty dt e^{-tE} \mathbb{E}_{\mathbb{P}^0 \times P} \left[g_\lambda \left(\int_0^t |V(X_s + x)| ds \right) \middle| X_t + x = y \right] p_t^\Psi(x - y)$$

is also jointly continuous in (x, y) and by setting $x = y$ we get

$$\text{Tr } F_\lambda(K_E) = \int_{\mathbb{R}^d} dx \int_0^\infty \frac{dt}{t} e^{-tE} \mathbb{E}_{\mathbb{P}^0 \times P} \left[G_\lambda \left(\int_0^t |V(X_s + x)| ds \right) \middle| X_t = 0 \right] p_t^\Psi(0),$$

where $G_\lambda(x) = xg_\lambda(x) = xe^{-\lambda x}$. Varying F_λ and g_λ while keeping their relationship unchanged, we can extend this formula to any strictly increasing $F : [0, \infty) \rightarrow [0, \infty)$ such that $F(x) = x \int_0^\infty e^{-y} g(xy) dy$, where g is a non-negative function on \mathbb{R} . With non-negative and lower semi-continuous $G(x) = xg(x)$ then we have

$$\text{Tr } F(K_E) = \int_{\mathbb{R}^d} dx \int_0^\infty dt \frac{e^{-tE}}{t} \mathbb{E}_{\mathbb{P}^0 \times P} \left[G \left(\int_0^t |V(X_s + x)| ds \right) \middle| X_t = 0 \right] p_t^\Psi(0).$$

This leads to the following result.

Theorem 5.1. (Lieb-Thirring bound) *Let Assumption 5.1 hold, F, G be any functions satisfying the conditions above, and G furthermore be convex. Then*

$$(5.6) \quad N_0(V) \leq \frac{1}{F(1)} \int_0^\infty \frac{ds}{s} G(s) \int_{\mathbb{R}^d} p_{s/|V(x)|}^\Psi(0) \mathbb{1}_{\{|V(x)|>0\}} dx,$$

where $p_{s/|V(x)|}^\Psi(0) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-s\Psi(|\xi|^2/2)/|V(x)|} d\xi$.

We note that the right hand side of (5.6) may be infinite, dependent on the choice of G .

From the formulae above we see that the LT bound depends on the diagonal part of the heat kernel p_t^Ψ . To make this expression more explicit we can use a rearrangement-type argument (for details see [8] and its references). The result is

Theorem 5.2.

$$(5.7) \quad N_0(V) \leq \frac{2^{\frac{3d}{2}+1} \pi^{\frac{d}{2}}}{d\Gamma(\frac{d}{2})F(1)} \int_0^\infty \frac{ds}{s} G(s) \int_{\mathbb{R}^d} dx \int_0^\infty \left(\Psi^{-1} \left(\frac{r|V(x)|}{s} \right) \right)^{d/2} e^{-r} dr.$$

In the case when $\Psi \in \mathcal{B}_0$ has a scaling property, we can derive a more explicit formula.

Corollary 5.1. *Suppose that $\Psi \in \mathcal{B}_0$ is strictly monotone increasing and the assumptions of Theorem 5.1 hold. In addition, assume that there exists $\gamma > 0$ such that $\Psi(au) = a^\gamma \Psi(u)$ for all $a, u \geq 0$. Then*

$$(5.8) \quad N_0(V) \leq A \int_{\mathbb{R}^d} (\Psi^{-1}(|V(x)|))^{d/2} dx,$$

$$\text{where } A = \frac{2^{\frac{3d}{2}+1} \pi^{\frac{d}{2}} \Gamma(\frac{d}{2\gamma} + 1)}{d \Gamma(\frac{d}{2}) F(1)} \int_0^\infty G(s) s^{-1-\frac{d}{2\gamma}} ds.$$

Instead of the scaling property suppose now that there exists $\lambda > 0$ such that $\Psi(u) \geq Cu^\lambda$ with a constant $C > 0$.

Corollary 5.2. *Suppose that $\Psi \in \mathcal{B}_0$ is strictly monotone increasing and the assumptions of Theorem 5.1 hold. If $\Psi(u) \geq Cu^\lambda$, then*

$$(5.9) \quad N_0(V) \leq A \int_{\mathbb{R}^d} |V(x)|^{d/2\lambda} dx,$$

$$\text{where } A = \frac{2^{\frac{3d}{2}+1} \pi^{\frac{d}{2}} C^{-1/\lambda}}{d \Gamma(\frac{d}{2}) F(1)} \int_0^\infty G(s) s^{-1-\frac{d}{2\lambda}} ds.$$

Example 5.1. In some special cases of Bernstein functions Ψ we can derive more explicit forms of the Lieb-Thirring inequality. For further examples see [8].

- (1) *Fractional Schrödinger operators.* An application of the Riesz potential and the Sobolev inequality shows that for $V \in L^{d/\alpha}(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ we have

$$\|(-\Delta)^{\alpha/4} f\|_2^2 \geq \frac{1}{C} \|f\|_{\frac{2d}{d-2\alpha}}^2 \geq \frac{1}{C} (|V|^{1/2} f, |V|^{1/2} f) \|V\|_{d/\alpha}^{-1},$$

and thus the quadratic form $Q(f, g) = ((-\Delta)^{\alpha/4} f, (-\Delta)^{\alpha/4} g) - (|V|^{1/2} f, |V|^{1/2} g)$ is bounded from below for all $0 < \alpha < 2$. Then, since $p_t^\Psi(0) = C(\alpha, d)t^{-d/\alpha}$, where $C(\alpha, d) = \frac{\alpha(S_{d-1})(d/\alpha)}{\alpha(2\pi)^d}$, we obtain

$$N_0(V) \leq L_{\alpha, d} \int_{\mathbb{R}^d} |V(x)|^{d/\alpha} dx$$

$$\text{with } L_{\alpha, d} = \frac{C(\alpha, d)}{F(1)} \int_0^\infty s^{-1-d/\alpha} G(s) ds.$$

(2) *Fractional relativistic Schrödinger operators.* In this case we have

$$\begin{aligned} & \|(-\Delta + m^{2/\alpha})^{\alpha/4} f\|_2^2 \\ & \geq \|(-\Delta)^{\alpha/4} f\|_2^2 \geq \frac{1}{C} \|f\|_{\frac{2d}{d-2\alpha}}^2 \geq \frac{1}{C} (|V|^{1/2} f, |V|^{1/2} f) \|V\|_{d/\alpha}^{-1}. \end{aligned}$$

Then with $V \in L^{d/\alpha}(\mathbb{R}^d) \cap L^{d/2}(\mathbb{R}^d)$, $m \neq 0$, we have

$$N_0(V) \leq L_{\alpha,d}^{(1)} \int_{\mathbb{R}^d} |V(x)|^{d/\alpha} dx + L_{\alpha,d}^{(2)} \int_{\mathbb{R}^d} |V(x)|^{d/2} dx,$$

where $L_{\alpha,d}^{(j)} = \frac{C_j(\alpha,d)}{F(1)} \int_0^\infty s^{-1-d/2} G(s) ds$, $j = 1, 2$, and with constants C_j which we can make explicit.

(3) *Jump-diffusion operators.* Choose the parameters to be $a = 1$ and $0 < b \leq 1$. Since in this case

$$p_t(x-y) \leq \left(t^{-d/2} \wedge (bt)^{-d/\alpha} \right) \wedge \left(t^{-d/2} e^{-|x-y|^2/ct} + (bt)^{-d/\alpha} \wedge \frac{bt}{|x-y|^{d+\alpha}} \right),$$

with $V \in L^{d/2}(\mathbb{R}^d) \cap L^{d/\alpha}(\mathbb{R}^d)$ we obtain

$$N_0(V) \leq L \int_{\mathbb{R}^d} |V(x)|^{d/2} dx + L_\alpha \int_{\mathbb{R}^d} |V(x)|^{d/\alpha} dx,$$

with $L = \frac{c}{F(1)} \int_0^\infty s^{-1-d/2} G(s) ds$ and $L_\alpha = \frac{c}{F(1)} \int_0^\infty s^{-1-d/\alpha} G(s) ds$, with an appropriate constant $c > 0$.

Acknowledgments: JL thanks BCAM, Bilbao, for an invitation to speak at the workshop *Fractional Calculus, Probability and Non-local Operators: Applications and Recent Developments*. JL is appreciative for collaboration on this and related subjects also with F. Hiroshima, T. Ichinose, J. Małecki over the years. KK was supported by the National Science Center (Poland) post-doctoral internship grant on the basis of the decision No. DEC-2012/04/S/ST1/00093.

REFERENCES

1. S.O. Durugo: A class of extended Airy functions of the first kind and spectral properties of the massless relativistic quartic anharmonic oscillator, PhD Thesis, Loughborough University, UK, 2014
2. S.O. Durugo and J. Lőrinczi: Higher order Airy functions of the first kind, preprint, 2014
3. S.O. Durugo and J. Lőrinczi: Spectral properties of the massless relativistic quartic oscillator, preprint, 2014

4. F. Hiroshima: Functional integral approach to semi-relativistic Pauli-Fierz models, to appear in *Adv. Math.*, 2014
5. F. Hiroshima, T. Ichinose and J. Lőrinczi: Path integral representation for Schrödinger operators with Bernstein functions of the Laplacian, *Rev. Math. Phys.* **24**, 1250013, 2012
6. F. Hiroshima, F., T. Ichinose and J. Lőrinczi: Probabilistic representation and fall-off of bound states of relativistic Schrödinger operators with spin 1/2, *Publ. RIMS* **49**, 189-214, 2013
7. F. Hiroshima and J. Lőrinczi: Functional integral representation of the Pauli-Fierz model with spin 1/2, *J. Funct. Anal.* **254**, 2127-2185, 2008
8. F. Hiroshima and J. Lőrinczi: Lieb-Thirring bound for Schrödinger operators with Bernstein functions of the Laplacian, *Commun. Stoch. Anal.* **6**, 589-602, 2013
9. F. Hiroshima, J. Lőrinczi and T. Takaesu: A probabilistic representation of the ground state expectation of fractional powers of the boson number operator, *J. Math. Anal. Appl.* **395**, 437-447, 2012
10. K. Kaleta and T. Kulczycki: Intrinsic ultracontractivity for Schrödinger operators based on fractional Laplacians, *Potential Anal.* **33**, 313-339, 2010
11. K. Kaleta and J. Lőrinczi: Fractional $P(\phi)_1$ -processes and Gibbs measures, *Stoch. Proc. Appl.* **122**, 3580-3617, 2012
12. K. Kaleta and J. Lőrinczi: Pointwise estimates of the eigenfunctions and intrinsic ultracontractivity-type properties of Feynman-Kac semigroups for a class of Lévy processes, to appear in *Ann. Probab.*, 2014
13. K. Kaleta and P. Sztonyk: Small time sharp bounds for kernels of convolution semigroups, preprint, 2014, available at arXiv:1403.0912
14. J. Lőrinczi, F. Hiroshima, F. and V. Betz: *Feynman-Kac-Type Theorems and Gibbs Measures on Path Space. With Applications to Rigorous Quantum Field Theory*, De Gruyter Studies in Mathematics **34**, 2011
15. J. Lőrinczi and J. Malecki: Spectral properties of the massless relativistic harmonic oscillator, *J. Diff. Equations* **253**, 2846-2871, 2012
16. R. Schilling, R. Song and Z. Vondraček: *Bernstein Functions. Theory and Applications*, De Gruyter Studies in Mathematics **37**, 2010