

# Generalized pathway entropy and its applications in diffusion entropy analysis and fractional calculus

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*Dedicated to Professor Francesco Mainardi on the occasion of his  
retirement*

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## Abstract

We presented background information about various entropies in the literature. The pathway idea of Mathai (2005) is shown to be inferable from the maximization of a certain generalized entropy measure and established connections to outstanding problems in astronomy and physics. In this paper we proved that the generalized entropy of Mathai associated with diffusive processes grows linearly with the logarithm of time, and the rate of growth is independent of the generalized entropy parameter. We also proposed some results concerning images of generalized Bessel function under the pathway operator and its special cases including some trigonometric functions. Situations are listed where a generalized entropy of order  $\alpha$  leads to pathway models, exponential and power law behavior and related differential equations.

*Keywords:* generalized entropy, standard deviation analysis, diffusion entropy analysis, fractional integral transform, generalized Bessel function, fractional reaction diffusion.

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## 1. Introduction.

The normal (Gaussian) distribution is a family of continuous probability distributions and is ubiquitous in the field of statistics and probability (Feller [0]). The importance of the normal distribution as a model of quantitative phenomena is due to the central limit theorem. The normal distribution maximizes Shannon entropy among all distributions with known mean

and variance and in information theory, Shannon entropy is the measure of uncertainty associated with a random variable.

In statistical mechanics, Gaussian (Maxwell-Boltzmann) distribution maximizes the Boltzmann-Gibbs entropy under appropriate constraints (Gell-Mann and Tsallis [0]). Given a probability distribution  $P = \{p_i\} (i = 1, \dots, m)$ , with  $p_i$  representing the probability of the system to be in the  $i$ th microstate, the Boltzmann-Gibbs entropy is

$$(1) \quad S = -k \sum_{i=1}^m p_i \ln p_i,$$

where  $k$  is the Boltzmann constant and  $m$  the total number of microstates. If all states are equally probable it leads to the Boltzmann principle  $S = k \ln W (m = W)$ . Boltzmann-Gibbs entropy is equivalent to Shannons entropy if  $k = 1$ . If we consider such a system in contact with a thermostat then we obtain the usual Maxwell-Boltzmann distribution for the possible states by maximizing the Boltzmann-Gibbs entropy  $S$  with the normalization and energy constraints. However, in nature many systems show distributions which differ from the Maxwell-Boltzmann distribution. These are usually systems with strong autocorrelations preventing the convergence to the Maxwell-Boltzmann distribution in the sense of the central-limit theorem. Well known examples in physics are: self gravitating systems, charged plasmas, Brownian particles in the presence of driving forces, and, more generally, non-equilibrium states of physical systems (Abe and Okamoto [0], Gell-Mann and Tsallis [0]). Then it is natural to ask the question of whether non-Maxwell-Boltzmannian distributions can also be obtained from a corresponding maximum entropy principle, considering a generalized form for the entropy. For this purpose, different forms were proposed, as for instance we are investigating the link between entropic functionals and the corresponding families of distributions in Mathai's pathway model and we come to the conclusion that this link is also important to physically analyze fractional reaction equations in terms of probability theory.

The entropy is the rigorous measure of lack of information. The following are some of the generalizations of Shannon  $S$ .

$$(2) \quad R = \frac{\ln(\sum_{i=1}^k p_i^\alpha)}{1 - \alpha}, \quad \alpha \neq 1, \alpha > 0, \text{ (Rényi entropy of order } \alpha \text{ of 1961),}$$

$$(3) \quad H = \frac{(\sum_{i=1}^k p_i^\alpha - 1)}{2^{1-\alpha} - 1}, \quad \alpha \neq 1, \alpha > 0, \text{ (Havrda-Charvát entropy of order } \alpha \text{ of 1967),}$$

$$(4) \quad T = \frac{(\sum_{i=1}^k p_i^\alpha)}{1 - \alpha}, \quad \alpha \neq 1, \alpha > 0, \text{ (Tsallis non-extensive entropy of order } \alpha \text{ of 1988),}$$

where  $p_i > 0, i = 1, \dots, k, p_1 + \dots + p_k = 1$ . When  $\alpha \rightarrow 1$  all the entropies of order  $\alpha$  described in (2) to (4) go to Shannon entropy  $S$ .

In physical situations when an appropriate density is selected, one procedure is the maximization of entropy. Mathai and Rathie [0] consider various generalizations of Shannon entropy measure and describe various properties including additivity, characterization theorem etc. Mathai et al. [0] introduced a new generalized entropy measure which is a generalization of the Shannon entropy measure. Applying the maximum entropy principle with normalization and energy constraints to Mathai's entropic functional, the corresponding parametric families of distributions of generalized type-1 beta, type-2 beta, generalized gamma, generalized Mittag-Leffler, and Lévy are obtained. For a multinomial population  $P = (p_1, \dots, p_k)$ ,  $p_i \geq 0, i = 1, \dots, k, p_1 + p_2 + \dots + p_k = 1$ , the Mathai's entropy measure is given by the relation

$$(5) \quad M_{k,\alpha}(P) = \frac{\sum_{i=1}^k p_i^{2-\alpha} - 1}{\alpha - 1}, \quad \alpha \neq 1, -\infty < \alpha < 2, \text{ (discrete case)}$$

$$(6) \quad M_\alpha(f) = \frac{1}{\alpha - 1} \left[ \int_{-\infty}^{\infty} [f(x)]^{2-\alpha} dx - 1 \right], \quad \alpha \neq 1, \alpha < 2 \text{ (continuous case).}$$

By optimizing Mathai's entropy measure, one can arrive at pathway model of Mathai [0], which consists of many of the standard distributions in statistical literature as special cases. For fixed  $\alpha$ , consider the optimization of  $M_\alpha(f)$ , which implies optimization of  $\int_x [f(x)]^{2-\alpha} dx$ , subject to the following conditions:  $f(x) \geq 0$  for all  $x$ ,  $\int_x f(x) dx < \infty$  and the following two moment-like expressions are fixed quantities for all functional  $f$ ,  $\int_x x^{\rho(1-\alpha)} f(x) dx$  is fixed for all  $f$ ,  $\int_x x^{\rho(1-\alpha)+\delta} f(x) dx$  is fixed for all  $f$ , where  $\rho$  and  $\delta$  are fixed parameters.

By using calculus of variation, one can obtain the Euler equation as

$$\begin{aligned} \frac{\partial}{\partial f} [f^{2-\alpha} - \lambda_1 x^{\rho(1-\alpha)} f + \lambda_2 x^{\rho(1-\alpha)+\delta} f] &= 0 \\ \Rightarrow (2 - \alpha) f^{1-\alpha} &= \lambda_1 x^{\rho(1-\alpha)} \left[ 1 - \frac{\lambda_2}{\lambda_1} x^\delta \right], \quad \alpha \neq 1, 2 \\ (7) \quad \Rightarrow f_1 &= c_1 x^\rho [1 - a(1 - \alpha)x^\delta]^{\frac{1}{1-\alpha}} \end{aligned}$$

for  $\frac{\lambda_2}{\lambda_1} = a(1 - \alpha)$ ,  $a > 0$  with  $\alpha < 1$  for type-1 beta,  $\alpha > 1$  for type-2 beta,  $\alpha \rightarrow 1$  for gamma, and  $\delta = 1$  for Tsallis statistics. For more details the reader may refer to the papers of Mathai et al. [0], Mathai and Haubold [0]. When  $\alpha \rightarrow 1$ , the Mathai's entropy measure  $M_\alpha(f)$  goes to the Shannon entropy measure and this is a variant of Havrda-Charvát entropy, and the variant form therein is Tsallis entropy. Then when  $\alpha$  increases from 1,  $M_\alpha(f)$  moves away from Shannon entropy. Thus  $\alpha$  creates a pathway moving from one function to another, through the generalized entropy also. This is the entropic pathway. One can derive Tsallis statistics and superstatistics (Beck [0], Beck and Cohen [0]) by using Mathai's entropy. It is shown that when the model is applied to physical situations then the current hot topics of Tsallis statistics and superstatistics in statistical mechanics become special cases of the pathway model, and the model is capable of capturing many stable situations as well as the unstable or chaotic neighborhoods of the stable situations and transitional stages.

In Section 2 we demonstrate that the extensive generalized entropy associated with diffusive processes grows linearly with the logarithm of time, and the rate of growth is independent of the extensive generalized entropy parameter. In Section 3 we proposed some results concerning images of generalized Bessel function under the pathway operator and its special cases including some trigonometric functions. In the last section we listed an example where a generalized entropy of order  $\alpha$  leads to pathway models, exponential and power law behavior and related differential equations.

## 2. Diffusion entropy analysis.

In this section we focus upon the scaling properties of a time series. By summing the terms of a time series we get a trajectory and the trajectory can be used to generate a diffusion process. There is scaling if, in the stationary condition, a diffusion process can be described by the following probability function (pdf):

$$(8) \quad p(x, t) = \frac{1}{t^\delta} F\left(\frac{x}{t^\delta}\right),$$

where  $x$  denotes the diffusion variable and  $p(x, t)$  is its pdf at time  $t$ . The coefficient  $\delta$  is called the scaling exponent. We define the scaling of a time series as the scaling exponent of a diffusion process generated by that time series. The purpose of the Diffusion entropy analysis (DEA) algorithm is to establish the possible existence of scaling, either normal or anomalous, in the most efficient way as possible without altering the data with any form of detrending.

Let us consider the simplifying assumption of considering large enough times as to make the continuous assumption valid. This method of analysis is based upon the evaluation of the Shannon entropy (continuous version) of the pdf of the diffusion process that reads

$$(9) \quad S(t) = - \int_{-\infty}^{+\infty} dx p(x, t) \ln p(x, t)$$

Using the scaling condition of (8) we obtain

$$(10) \quad S(t) = A + \delta \ln(t), \quad A = - \int_{-\infty}^{+\infty} dy F(y) \ln F(y),$$

where  $y = \frac{x}{t^\delta}$ , for more details see [0], [0]. Equation (10) indicates that in the case of a diffusion process with a scaling pdf, its entropy  $S(t)$  increases linearly with  $\ln(t)$ . Numerically, the scaling exponent  $\delta$  can be evaluated by using fitting curves with the function of the form  $f_S(t) = \kappa + \delta \ln(t)$  that, when graphed on linear-log graph paper, yields straight lines.

Hence the DEA provides a better way to detect  $\delta$  correctly. It is so because DEA analyzes directly the pdf of the diffusion processes, without using the moments of the distribution. Instead, all the other methods used for detecting scaling Variance Scaling Analysis, Hurst R/S Analysis, Detrended Fluctuation Analysis, Relative Dispersion Analysis, Spectral Analysis, Spectral Wavelet Analysis are subtly based on the Gaussian assumption and, so, upon a variance that can be used to monitor scaling. In the variance based methods, scaling is studied by direct evaluation of the time behavior of the variance of the diffusion process. If the variance scales, one would have

$$\sigma_x^2(t) \sim t^{2H},$$

where  $H$  is the Hurst exponent in honor of Hurst [0]. The problem is that the scaling detected by the variance methods,  $H$ , may not exist or may not coincide with the correct scaling,  $\delta$ . If the time series is characterized by what Mandelbrot called Fractional Brownian Motion, we have  $H = \delta$ . Consequently, the scaling of this type of noise can be detected by using the variance methods. If, on the contrary, the time series is characterized, for example, by Lévy properties ([0], [0]),  $H \neq \delta$  and the variance methods cannot be used to detect the true scaling. A diffusion process generated by Lévy walk is characterized by the relation

$$(11) \quad \delta = \frac{1}{3 - 2H}.$$

In the case of Lévy flights, the exponent  $H$  cannot be determined because the variance diverges, whereas the scaling  $\delta$  exists and can be determined

by using the diffusion entropy analysis. The above conclusions suggest that to determine the real statistical properties of a time series it is not enough to study the scaling with only one type of analysis. Only the joint use of two scaling analysis methods, the variance scaling analysis and the diffusion entropy analysis, can determine the real nature, Gauss or Lévy or something else, of a time series. We have to determine  $H$  and  $\delta$ . Then, if  $H = \delta$  we can conclude that fractional Brownian noise may characterize the signal. If, instead,  $H \neq \delta$  we have to look for a different type of noise. If we find that the relation (11) holds true, we can have good reasons to conclude that the noise is characterized by Lévy statistics. Moreover, DEA may be used for studying the transition from the dynamics to the thermodynamics of the diffusion process.

### 2.1. Mathai's entropy.

The entropies defined in (3), (4) and (5) are non-additive. An additive form of (5) is defined as follows: For a multinomial population  $P = (p_1, \dots, p_k)$ ,  $p_i \geq 0$ ,  $i = 1, \dots, k$ ,  $p_1 + p_2 + \dots + p_k = 1$ , the Mathai's extensive generalized entropy measure is given by the relation

$$(12) \quad M_{k,\alpha}^*(P) = \frac{\ln(\sum_{i=1}^k p_i^{2-\alpha})}{\alpha - 1}, \quad \alpha \neq 1, \quad -\infty < \alpha < 2, \quad (\text{discrete case})$$

As can be expected, when a logarithmic function is involved, as in the case of (1), (2) and (12) the entropy is additive. The DEA performs better than the other methods of analysis due to the fact that the information extracted from the pdf, expressed under the form of entropy, is larger than the information extracted from the pdf variance. Note that the entropy indicator need not to be the Shannon indicator. We hope that to detect scaling extensive generalized entropy defined in (12) is as effective as the Shannon entropy.

In the continuum limit:  $p_i \approx p(x, t)\Delta x$ , where  $x$  is the random variable, for example displacement for random walker, and  $p(\cdot)$  is the probability function. We consider Brownian and anomalous diffusive processes characterized by a probability function given in (8). The normalization of probabilities implies:  $\int f(x)dx = 1$ . A Brownian process is characterized by the lack of time correlations and has the probability function:

$$p(x, t) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}},$$

where  $D$  is the diffusion constant. In the subdiffusive regime,  $0 < \delta < \frac{1}{2}$ , there are negative correlations or antipersistence, while in the superdiffusive

regime (Lévy flights),  $\frac{1}{2} < \delta < 1$ , there are positive time correlations or persistence.

The time dependence of entropy for anomalous diffusion processes is obtained by substituting  $p_i \approx p(x, t)\Delta x$ , into (12), replacing the sum by an integral and using (8) we have

$$M_{k,\alpha}^*(P, t) = \frac{\ln(\sum_{i=1}^k p_i^{2-\alpha})}{\alpha - 1} \approx \frac{1}{\alpha - 1} \ln \left[ (\Delta x)^{1-\alpha} \int (p(x, t))^{2-\alpha} dx \right]$$

Using simple algebra we will get the associated extensive generalized entropy in the following form

$$(13) \quad M_{k,\alpha}^*(P, t) = -\ln \Delta x - \frac{1}{1-\alpha} \ln \left[ \int (f(y))^{2-\alpha} dy \right] + \delta \ln t.$$

where  $y = \frac{x}{t^\delta}$ . First we note that Equation (13) is precisely analogous to the logarithmic time evolution in (10). Hence  $M_{k,\alpha}^*(P, t) = a(\alpha) \ln(t) + b(\alpha)$ , with  $a(\alpha) = \delta$  independent of the extensive generalized entropy parameter  $\alpha$ .

The non-stationary dynamical transient may be simulated by a non stationary pdf of the type

$$(14) \quad p(x, t) = \frac{1}{t^{\delta(t)}} F\left(\frac{x}{t^{\delta(t)}}\right),$$

where the pdf scaling exponent  $\delta(t)$  changes with the diffusion time  $t$ . Let us suppose that

$$(15) \quad \delta(t) = \delta_0 + \eta \ln t.$$

Since the scaling parameter  $\delta$  cannot exceed the ballistic value  $\delta = 1$  in the case of a dynamical approach to diffusion with fluctuation of limited intensity, this condition applies to the time scale defined by

$$\eta \ln t < 1 - \delta_0.$$

We notice that in the new non-stationary condition the traditional entropy indicator yields

$$(16) \quad M_{k,\alpha}^*(P, \tau) = b(\alpha) + \delta_0 \tau + \eta \tau^2,$$

where  $\tau = \ln t$ . Furthermore, the benefits stemming from the entropic method of analysis of a diffusion process (the DEA) are not limited to the detection of the true asymptotic scaling  $\delta$ . We can explore the still unknown regime of transition from dynamics to thermodynamics, and we can also address the ambitious issue of studying the time series produced by non-stationary processes.

## 2.2. Diffusion Entropy Analysis Based on Non-extensive Mathai's entropy.

A diffusion entropy analysis study based on non-extensive Tsallis indicator can be obtained in [0]. The non-extensive Tsallis indicator corresponding to the continuous formalism is given by

$$(17) \quad T_\alpha(t) = \frac{1}{\alpha - 1} \left[ 1 - \int_{-\infty}^{\infty} dx [p(x, t)]^\alpha \right].$$

The quadratic form of (16) suggests that the choice of  $\delta(t)$  given by (15) has the mathematical meaning of the quadratic term in the Taylor expansion of the diffusion entropy (9). As a consequence, we should expect that, in general,  $\delta(t)$  always assumes the form of (15), at least for small values of  $\ln t$ . The non-extensive Mathai's entropy [0] reads in the continuous formalism

$$(18) \quad M_\alpha(t) = \frac{1}{\alpha - 1} \left[ \int_{-\infty}^{\infty} dx [p(x, t)]^{2-\alpha} - 1 \right], \quad \alpha \neq 1, \quad \alpha < 2 \text{ (continuous case).}$$

It is straightforward to prove that this entropic indicator coincides with that of (9) in the limit where the entropic index  $\alpha \rightarrow 1$ . Let us make the assumption that in the diffusion regime, the departure from this traditional value is weak, and assume  $\epsilon \equiv \alpha - 1 \ll 1$ . This allows us to use the following approximate expression for the non-extensive entropy

$$M_\alpha(t) = - \int_{-\infty}^{+\infty} dx p(x, t) \ln p(x, t) + \frac{\epsilon}{2} \int_{-\infty}^{+\infty} dx p(x, t) [\ln p(x, t)]^2.$$

In the specific case where the nonscaling condition of (14) applies, this entropy yields the form

$$M_\alpha = A + \epsilon B + (1 + \epsilon A) \delta(t) \ln t + \frac{\epsilon}{2} [\delta(t) \ln t]^2,$$

where  $A$  and  $B$  are two constants related to  $F(y)$  of (14). These theoretical remarks demonstrate that this non-extensive approach to the diffusion entropy makes it possible to detect the strength of the deviation from the steady condition. In fact it could be proved that  $\epsilon = 0$  implies a steady condition. The conclusion of this section is that the breakdown of the scaling property of (8) can be revealed by the DEA under the form of an entropic index  $\alpha$  departing from the condition of ordinary statistical mechanics, namely  $\alpha = 1$ .

Figure 1(a) and (b) show the curves corresponding to  $\alpha = 0.8, 1, 1.2$  respectively with the non-extensive Tsallis and Mathai's  $\alpha$ -entropy indicator



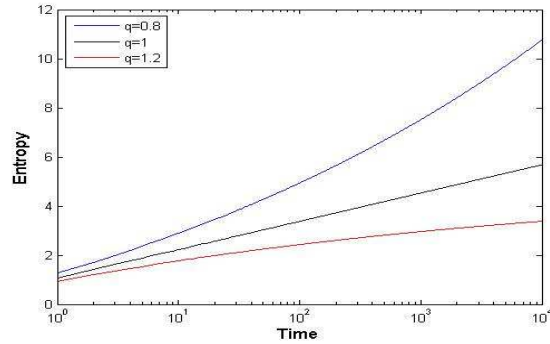


Figure 1. Diffusion Entropy by using the non-extensive Tsallis entropy (17)

as a function of time  $t$  applied to the following Brownian diffusion equation

$$p(x, t) = \frac{1}{\sqrt{\pi t}} e^{-\frac{x^2}{t}}.$$

By adopting the non-extensive Tsallis entropy (17), we get

$$(19) \quad S_{\alpha}(t) = \begin{cases} \frac{(1-\pi^{\frac{1-\alpha}{2}} \alpha^{-\frac{1}{2}} t^{\frac{1-\alpha}{2}})}{\alpha^{-1}} & \alpha \neq 1 \\ \frac{1}{2} + \frac{1}{2} \ln(\pi t) & \alpha = 1, \end{cases}$$

and by using the non-extensive Mathai's entropy (18), we get

$$(20) \quad M_{\alpha}(t) = \begin{cases} \frac{(\pi^{\frac{\alpha-1}{2}} (2-\alpha)^{-\frac{1}{2}} t^{\frac{\alpha-1}{2}} - 1)}{\alpha^{-1}} & \alpha \neq 1, \alpha < 2 \\ \frac{1}{2} + \frac{1}{2} \ln(\pi t) & \alpha = 1. \end{cases}$$

Figures show the effect of the entropic index  $\alpha$  upon the entropy of the diffusion. The entropic index  $\alpha$  has the effect to bend the curve. In Figure 1 if  $\alpha > 1$  the curve becomes more convex and if  $\alpha < 1$  the curve becomes more concave. But in Figure 2 we can see the exact reverse behaviour.

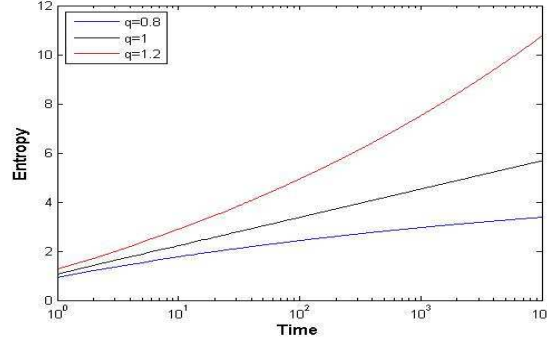


Figure 2. Diffusion Entropy by using the non-extensive Mathai's entropy (18)

### 3. Pathway integral operator of generalized Bessel function and their special cases.

By using the pathway idea of Mathai [0], a pathway fractional integral operator (pathway operator) is defined by Nair [0] and is defined as follows: Let  $f(x) \in L(a, b)$ ,  $\eta \in C$ ,  $\Re(\eta) > 0$ ,  $a > 0$  and  $\alpha < 1$ , then

$$(21) \quad (P_{0+}^{(\eta, \alpha)} f)(x) = x^{\eta-1} \int_0^{\frac{x}{a(1-\alpha)}} \left[ 1 - \frac{a(1-\alpha)t}{x} \right]^{\frac{\eta}{(1-\alpha)}-1} f(t) dt,$$

where  $\alpha$  is the pathway parameter and  $f(t)$  is an arbitrary function. In the pathway model, as  $\alpha \rightarrow 1$ , we can see that

$$\lim_{\alpha \rightarrow 1-} [1 - a(1-\alpha)x^\delta]^{\frac{\eta}{1-\alpha}} = \lim_{\alpha \rightarrow 1+} [1 + a(\alpha-1)x^\delta]^{-\frac{\eta}{\alpha-1}} = e^{-a\eta x^\delta}.$$

When  $\alpha \rightarrow 1-$ ,  $[1 - \frac{a(1-\alpha)t}{x}]^{\frac{\eta}{1-\alpha}} \rightarrow e^{-\frac{a\eta}{x}t}$ . Thus the operator will become

$$P_{0+}^{\eta, 1} = x^{\eta-1} \int_0^\infty e^{-\frac{a\eta}{x}t} f(t) dt = x^{\eta-1} L_f\left(\frac{a\eta}{x}\right),$$

the Laplace transform of  $f$  with parameter  $\frac{a\eta}{x}$ . When  $\alpha = 0, a = 1$  in (21) the integral will become,

$$P_{0+}^{\eta,0} = \Gamma(\eta)I_{0+}^{\eta} = \int_0^x (x-t)^{\eta-1}f(t)dt,$$

where  $I_{0+}$  is the left-sided Riemann-Liouville fractional integral operator defined for  $\eta \in \mathbb{C}, x > 0$ , (Samko et al, [0]) as:

$$(22) \quad (\mathcal{I}_{0+}^{\eta}f)(x) \equiv \frac{1}{\Gamma(\eta)} \int_0^x (x-t)^{\eta-1}f(t)dt \quad \Re(\eta) > 0.$$

It is also observed that when the pathway parameter,  $\alpha = 0, a = 1$  and  $f(t)$  replaced by  ${}_2F_1(\nu + \beta, -\eta; \nu; 1 - \frac{t}{x})f(t)$  then the pathway operator yields to

$$\int_0^x (x-t)^{\alpha-1} {}_2F_1\left(\nu + \beta, -\eta; \nu; 1 - \frac{t}{x}\right) f(t)dt = \frac{\Gamma(\nu)}{x^{-\nu-\beta}} I_{0+}^{\nu,\beta,\eta},$$

where  $I_{0+}^{\nu,\beta,\eta}$  denotes the Saigo fractional integral operator, [0]. When  $\alpha \rightarrow 1, \eta = 1$  and replace  $f(t)$  by  $t^{\beta-1}{}_0F_1(; \beta; \delta t)$  in pathway fractional integral operator then we are essentially dealing with distribution functions under a gamma Bessel type model in a practical statistical problem, see [0]. Hence a connection between statistical distribution theory and fractional calculus is established so that one can make use of the rich results in statistical distribution theory for further development of fractional calculus and vice versa. The pathway fractional integral operator has found applications in reaction-diffusion problems, non-extensive statistical mechanics, non-linear waves, fractional differential equations, non-stable neighborhoods of physical system etc.

Our goal is to study in general the pathway fractional integration of the Bessel functions, the modified Bessel functions, the spherical Bessel functions and the modified spherical Bessel functions together. For this we consider the linear differential equation

$$(23) \quad z^2w''(z) + b zw'(z) + (cz^2 + d)w(z) = 0,$$

where  $b, c \in \mathbb{C}$  and  $d = d_1p^2 + d_2p + d_3$ , with  $d_1, d_2, d_3, p \in \mathbb{C}$ . By putting  $d_1 = -1, d_2 = 1 - b$  and  $d_3 = 0$ , and the differential equation (23) is

$$(24) \quad z^2w''(z) + b zw'(z) + (cz^2 - p^2 + (1-b)p)w(z) = 0.$$

We obtain a particular solution of (24) for all  $z \in \mathbb{C}, z \neq 0$ , and  $b, c, p \in \mathbb{C}$  by [0],

$$(25) \quad W_{p,b,c}(z) = \sum_{k=0}^{\infty} \frac{(-c)^k}{k! \Gamma(p + \frac{b+1}{2} + k)} \left(\frac{z}{2}\right)^{\nu+2k},$$

where  $W_{p,b,c}(\cdot)$  is the generalized Bessel function of the first kind and which permits the study of Bessel, modified Bessel, spherical Bessel and modified spherical Bessel functions together. It is clear that for  $c = 1$  and  $b = 1$  the function  $W_{p,b,c}$  reduces to  $J_p$ , Bessel function of the first kind of order  $p$ , when  $c = -1$  and  $b = 1$  the function  $W_{p,b,c}$  becomes  $I_p$ , modified Bessel function of the first kind of order  $p$ . Similarly, when  $c = 1$  and  $b = 2$  the function  $W_{p,b,c}$  reduces to  $2j_p/\sqrt{\pi}$  where  $j_p$  is the spherical Bessel function of order  $p$ , while if  $c = -1$  and  $b = 2$ , then  $W_{p,b,c}$  becomes  $2i_p/\sqrt{\pi}$ , where  $i_p$  is the modified spherical Bessel function of order  $p$ . Further, from (25) we have  $W_p(0) = 0$ .

Our main result in this section is based on the preliminary assertion giving composition formula of pathway fractional integral operator (21) with a power function.

**Lemma 3.1.** [ [0], Lemma 1] Let  $\rho, \eta \in \mathbb{C}, \Re(\eta) > 0$  and  $\alpha < 1$  such that

$$(26) \quad \Re(\rho) > 0, \quad \Re\left(\frac{\eta}{1-\alpha}\right) > -1.$$

Then

$$(27) \quad (P_{0+}^{(\eta,\alpha)} t^{\rho-1})(x) = \frac{\Gamma(\rho)\Gamma(1 + \frac{\eta}{1-\alpha})}{\Gamma(\frac{\eta}{1-\alpha} + \rho + 1)} \frac{x^{\eta+\rho}}{[a(1-\alpha)]^\rho}, \quad x > 0.$$

In particular, for  $x > 0$

$$(28) \quad \lim_{\alpha \rightarrow 1-} (P_{0+}^{(\eta,\alpha)} t^{\rho-1})(x) \rightarrow \Gamma(\rho) \frac{x^{\eta+\rho}}{[a\eta]^\rho}.$$

We prove that such compositions are expressed in terms of the generalized Wright hypergeometric function  ${}_p\Psi_q(z)$  defined for  $z \in \mathbb{C}$ , complex  $a_i, b_j \in \mathbb{C}$ , and real  $\alpha_i, \beta_j \in \mathbb{R} (i = 1, 2, \dots, p; j = 1, 2, \dots, q)$  by the series

$$(29) \quad {}_p\Psi_q(z) = {}_p\Psi_q\left[\begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \middle| z\right] \equiv \sum_{k=0}^{\infty} \frac{\prod_{i=1}^p \Gamma(a_i + \alpha_i k)}{\prod_{j=1}^q \Gamma(b_j + \beta_j k)} \frac{z^k}{k!}$$

Asymptotic behavior of this function for large values of argument of  $z$  was investigated by Fox [0] and Wright [0] under the condition

$$(30) \quad \sum_{j=1}^q \beta_j - \sum_{i=1}^p \alpha_i > -1.$$

Under this conditions  ${}_p\Psi_q(z)$  is an entire function, see Fox [0].

The following assertion is based on the corresponding statement for the generalized fractional integral (21) obtained in [0].

**Theorem 3.1.** *Let  $\eta, \rho, b, p, c \in \mathbb{C}$ ,  $\Re(1 + \frac{\eta}{1-\alpha}) > 0$ ,  $\Re(\rho + p) > 0$ ,  $\Re(\eta) > 0$ ,  $\alpha < 1$  and  $P_{0+}^{(\eta, \alpha)}$  be the pathway fractional integral. Then there holds the image.*

$$(31) \quad \left( P_{0+}^{(\eta, \alpha)} t^{\rho-1} W_{p,b,c}(t) \right) (x) = \frac{x^{p+\rho+\eta} \Gamma(1 + \frac{\eta}{1-\alpha})}{2^p [a(1-\alpha)]^{p+\rho}} \times {}_1\Psi_2 \left[ \begin{matrix} (p+\rho, 2) \\ (\kappa, 1), (\frac{\eta}{1-\alpha} + p + \rho + 1, 2) \end{matrix} \middle| - \frac{cx^2}{4[a(1-\alpha)]^2} \right],$$

where  ${}_p\Psi_q(z)$  is given by (29) and  $\kappa = p + \frac{b+1}{2}$ .

**Proof.** An application of integral operator (21) to the generalized Bessel function (25) leads to the formula

$$(32) \quad \left( P_{0+}^{(\eta, \alpha)} t^{\rho-1} W_{p,b,c}(t) \right) (x) = \left( P_{0+}^{(\eta, \alpha)} \sum_{k=0}^{\infty} \frac{(-c)^k (1/2)^{\nu+2k}}{k! \Gamma(\kappa + k)} (t)^{\rho+p+2k-1} \right) (x)$$

Now changing the orders of integration and summation in the right hand side of (32) yields

$$(33) \quad \left( P_{0+}^{(\eta, \alpha)} t^{\rho-1} W_{p,b,c}(t) \right) (x) = \sum_{k=0}^{\infty} \frac{(-c)^k (1/2)^{\nu+2k}}{k! \Gamma(\kappa + k)} \left( P_{0+}^{(\eta, \alpha)} (t)^{\rho+p+2k-1} \right) (x)$$

Note that for any  $k = 0, 1, \dots$ ,  $\Re(\rho + p + 2k) \geq \Re(\rho + p) > 0$ . Applying Lemma 3.1 and replacing  $\rho$  by  $\rho + p + 2k$ , we obtain

$$(34) \quad \left( P_{0+}^{(\eta, \alpha)} t^{\rho-1} W_{p,b,c}(t) \right) (x) = \frac{x^{p+\rho+\eta} \Gamma(1 + \frac{\eta}{1-\alpha})}{2^p [a(1-\alpha)]^{p+\rho}} \times \sum_{k=0}^{\infty} \frac{\Gamma(\rho + p + 2k)}{\Gamma(\kappa + k) \Gamma(\frac{\eta}{1-\alpha} + \rho + p + 1 + 2k)} \frac{(-cx^2)^k}{[4a^2(1-\alpha)^2]^k k!}$$

Interpreting the right hand side of (34), the equality (31) can be obtained from by using the definition of generalized Wright function.  $\square$

**Remark 3.1.** For  $\alpha \rightarrow 1-$ , (31) gives the Laplace transform image:

$$\lim_{\alpha \rightarrow 1-} \left( P_{0+}^{(\eta, \alpha)} t^{\rho-1} W_{p,b,c}(t) \right) (x) = \frac{x^{p+\rho+\eta}}{2^p (a\eta)^{p+\rho}} {}_1\Psi_1 \left[ \begin{matrix} (p+\rho, 2) \\ (\kappa, 1) \end{matrix} \middle| - \frac{cx^2}{4(a\eta)^2} \right].$$

**Proof.** The Stirling's approximation for a gamma function, namely,

$$\Gamma(z + \beta) \approx 2\pi^{\frac{1}{2}}(z)^{z+\beta-\frac{1}{2}}e^{-z}$$

for  $|z| \rightarrow \infty$  and  $\beta$  a bounded quantity. In (31) when  $\alpha \rightarrow 1_-, \frac{\eta}{1-\alpha} \rightarrow \infty$  and using the Stirling's formula of gamma functions gives the result.  $\square$

### 3.1. Fractional integration of trigonometric functions.

For all  $b \in \mathbb{C}, p = -\frac{b}{2}$  then the generalized Bessel function  $W_{p,b,c}(z)$  in (25) coincides with the cosine function and hyperbolic cosine functions respectively by,

$$(35) \quad W_{-\frac{b}{2},b,c^2}(z) = \left(\frac{2}{z}\right)^{\frac{b}{2}} \frac{\cos(cz)}{\sqrt{\pi}} \text{ and } W_{-\frac{b}{2},b,-c^2}(z) = \left(\frac{2}{z}\right)^{\frac{b}{2}} \frac{\cosh(cz)}{\sqrt{\pi}}.$$

Similarly for all  $b \in \mathbb{C}, p = 1 - \frac{b}{2}$ , then the generalized Bessel function  $W_{p,b,c}(z)$  have the form

$$(36) \quad W_{1-\frac{b}{2},b,c^2}(z) = \left(\frac{2}{z}\right)^{\frac{b}{2}} \frac{\sin(cz)}{\sqrt{\pi}} \text{ and } W_{1-\frac{b}{2},b,-c^2}(z) = \left(\frac{2}{z}\right)^{\frac{b}{2}} \frac{\sinh(cz)}{\sqrt{\pi}}.$$

From Theorem 3.1 we obtain the following result:

**Corollary 3.1.** Let  $\eta, \rho, b, c \in \mathbb{C}, \Re(1 + \frac{\eta}{1-\alpha}) > 0, \Re(\rho) > 0, \Re(\eta) > 0, \alpha < 1$  and  $P_{0+}^{(\eta,\alpha)}$  be the pathway fractional integral. Then there holds the formula

$$(37) \quad \begin{aligned} \left(P_{0+}^{(\eta,\alpha)} t^{\rho-1} \cos(ct)\right)(x) &= \frac{\sqrt{\pi} x^{\rho+\eta} \Gamma(1 + \frac{\eta}{1-\alpha})}{[a(1-\alpha)]^\rho} \\ &\times {}_1\Psi_2 \left[ \begin{matrix} (\rho, 2) \\ (\frac{1}{2}, 1), (\frac{\eta}{1-\alpha} + \rho + 1, 2) \end{matrix} \middle| -\frac{c^2 x^2}{4[a(1-\alpha)]^2} \right], \end{aligned}$$

and

$$(38) \quad \begin{aligned} \left(P_{0+}^{(\eta,\alpha)} t^{\rho-1} \cosh(ct)\right)(x) &= \frac{\sqrt{\pi} x^{\rho+\eta} \Gamma(1 + \frac{\eta}{1-\alpha})}{[a(1-\alpha)]^\rho} \\ &\times {}_1\Psi_2 \left[ \begin{matrix} (\rho, 2) \\ (\frac{1}{2}, 1), (\frac{\eta}{1-\alpha} + \rho + 1, 2) \end{matrix} \middle| \frac{c^2 x^2}{4[a(1-\alpha)]^2} \right]. \end{aligned}$$

**Remark 3.2.** For  $\alpha \rightarrow 1_-$ , (37) and (38) reduces to

$$\lim_{\alpha \rightarrow 1_-} \left(P_{0+}^{(\eta,\alpha)} t^{\rho-1} \cos(ct)\right)(x) \rightarrow \frac{\sqrt{\pi} x^{\rho+\eta}}{(a\eta)^\rho} {}_1\Psi_1 \left[ \begin{matrix} (\rho, 2) \\ (\frac{1}{2}, 1) \end{matrix} \middle| -\frac{c^2 x^2}{4(a\eta)^2} \right],$$

and

$$\lim_{\alpha \rightarrow 1_-} \left(P_{0+}^{(\eta,\alpha)} t^{\rho-1} \cosh(ct)\right)(x) \rightarrow \frac{\sqrt{\pi} x^{\rho+\eta}}{(a\eta)^\rho} {}_1\Psi_1 \left[ \begin{matrix} (\rho, 2) \\ (\frac{1}{2}, 1) \end{matrix} \middle| \frac{c^2 x^2}{4(a\eta)^2} \right].$$

Thus from Theorem 3.1, the composition of Pathway fractional integral operators respectively with sine and hyperbolic sine functions can be obtained.

**Corollary 3.2.** Let  $\eta, \rho, b, c \in \mathbb{C}$ ,  $\Re(1 + \frac{\eta}{1-\alpha}) > 0$ ,  $\Re(\rho) > 0$ ,  $\Re(\eta) > 0$ ,  $\alpha < 1$  and  $P_{0+}^{(\eta, \alpha)}$  be the pathway fractional integral. Then there holds the formula

$$(39) \quad \begin{aligned} \left( P_{0+}^{(\eta, \alpha)} t^{\rho-1} \sin(ct) \right) (x) &= \frac{\sqrt{\pi}}{2} \frac{x^{\rho+\eta+1} \Gamma(1 + \frac{\eta}{1-\alpha})}{[a(1-\alpha)]^{\rho+1}} \\ &\times {}_1\Psi_2 \left[ \begin{matrix} (\rho+1, 2) \\ (\frac{3}{2}, 1), (\frac{\eta}{1-\alpha} + \rho + 2, 2) \end{matrix} \middle| -\frac{c^2 x^2}{4[a(1-\alpha)]^2} \right], \end{aligned}$$

and

$$(40) \quad \begin{aligned} \left( P_{0+}^{(\eta, \alpha)} t^{\rho-1} \sinh(ct) \right) (x) &= \frac{\sqrt{\pi}}{2} \frac{x^{\rho+\eta+1} \Gamma(1 + \frac{\eta}{1-\alpha})}{[a(1-\alpha)]^{\rho+1}} \\ &\times {}_1\Psi_2 \left[ \begin{matrix} (\rho+1, 2) \\ (\frac{3}{2}, 1), (\frac{\eta}{1-\alpha} + \rho + 2, 2) \end{matrix} \middle| \frac{c^2 x^2}{4[a(1-\alpha)]^2} \right]. \end{aligned}$$

**Remark 3.3.** For  $\alpha \rightarrow 1_-$ , (39) and (40) reduces to

$$\lim_{\alpha \rightarrow 1_-} \left( P_{0+}^{(\eta, \alpha)} t^{\rho-1} \sin(ct) \right) (x) \rightarrow \frac{\sqrt{\pi}}{2} \frac{x^{\rho+\eta+1}}{(a\eta)^{\rho+1}} {}_1\Psi_1 \left[ \begin{matrix} (\rho+1, 2) \\ (\frac{3}{2}, 1) \end{matrix} \middle| -\frac{c^2 x^2}{4[a\eta]^2} \right]$$

and

$$\lim_{\alpha \rightarrow 1_-} \left( P_{0+}^{(\eta, \alpha)} t^{\rho-1} \sinh(ct) \right) (x) \rightarrow \frac{\sqrt{\pi}}{2} \frac{x^{\rho+\eta+1}}{(a\eta)^{\rho+1}} {}_1\Psi_1 \left[ \begin{matrix} (\rho+1, 2) \\ (\frac{3}{2}, 1) \end{matrix} \middle| \frac{c^2 x^2}{4[a\eta]^2} \right].$$

**Example 3.1.** We consider special case of (37), which gives the result in terms of the Mittag-Leffler function. When  $\alpha = 0$ ,  $a = 1$  and replace  $\eta$  by  $\eta - 1$  we have

$$(41) \quad \left( I_{0+}^{\eta} t^{\rho-1} \cos(ct) \right) (x) = \sqrt{\pi} x^{\rho+\eta-1} {}_1\Psi_2 \left[ \begin{matrix} (\rho, 2) \\ (\frac{1}{2}, 1), (\eta + \rho, 2) \end{matrix} \middle| -\frac{c^2 x^2}{4} \right].$$

For  $\rho = c = 1$ , relation (41) takes the form

$$\left( I_{0+}^{\eta} \cos t \right) (x) = \pi^{\frac{1}{2}} x^{\eta} {}_1\Psi_2 \left[ \begin{matrix} (1, 2) \\ (\frac{1}{2}, 1), (1 + \eta, 2) \end{matrix} \middle| -\frac{x^2}{4} \right]$$

Applying the formulas

$$(42) \quad \Gamma\left(\frac{1}{2}\right) = \pi^{1/2}, \quad (2k)! = 4^k k! \left(\frac{1}{2}\right)_k, \quad k \in \mathbb{N}_0,$$

we find

$$(I_{0+}^{\eta} \cos t)(x) = x^{\eta} E_{2,1+\eta}(-x^2),$$

where  $E_{\alpha,\beta}(\cdot)$  denotes the two-index Mittag-Leffler function and which is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta + \alpha k)}, z \in \mathcal{C}.$$

For positive integer  $\eta = m \in \mathbb{N}$ ,

$$(I_{0+}^m \cos t)(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+m}}{(2k+m)!}, \quad m \in \mathbb{N}.$$

In particular, for  $m = 1$  we obtain the well known formula

$$(I_{0+}^1 \cos t)(x) \equiv \int_0^x \cos t \, dt = \sin x.$$

One can obtain similar kind of results in other cases too.

Thus these results are useful to derive certain composition formula involving Riemann-Liouville, Erdélyi-Kober, Saigo and pathway fractional operators on Bessel, modified Bessel, and spherical Bessel function of first kind. Particular attention is devoted to the technique of Laplace transform for treating these operators in a way accessible to applied scientists, avoiding unproductive generalities and excessive mathematical rigor.

#### 4. Applications.

For the sake of completeness we discuss the following application listed from Mathai and Haubold [0].

##### 4.1. Reaction models.

Reaction and relaxation processes in thermonuclear plasmas are governed by ordinary differential equations of the type

$$(43) \quad \frac{d}{dt} N(t) = -cN(t)$$

for exponential behavior. The quantity  $c$  is a thermonuclear function which is governed by the average of the Gamow penetration factor over the Maxwell-Boltzmannian velocity distribution of reacting species and has been extended to incorporate more general distributions than the normal distribution ([0]). To address non-exponential properties of a reaction or



relaxation process, the first-order time derivative can be replaced formally by a derivative of fractional order in the following way ( [0])

$$(44) \quad N(t) = N_0 - c^\nu {}_0D_t^{-\nu} N(t), \quad \nu > 0,$$

where  ${}_0D_t^{-\nu} f(t)$  is the Riemann-Liouville fractional integral operator, and the solution can be written in terms of Mittag-Leffler function:

$$(45) \quad N(t) = N_0 \sum_{k=0}^{\infty} \frac{(-1)^k [(ct)^\nu]^k}{\Gamma(1 + k\nu)} = N_0 E_\nu(-c^\nu t^\nu).$$

The Laplace transform of  $N(t)$  coming from (44) is

$$L_{N(t)}(s) = \frac{N_0}{s[1 + (\frac{c}{s})^\nu]},$$

which is a special case of general class of Laplace transforms associated with  $\alpha$ -Laplace stochastic processes and geometrically infinitely divisible statistical distributions.

Considering  $c$  to be a random variable itself,  $N(t)$  is to be taken as  $N(t|c)$  and can be written as

$$N(t|c) = N_0 t^{\mu-1} E_{\nu,\nu}^{\gamma+1}(-c^\nu t^\nu), \quad \gamma, \mu, \nu > 0,$$

which represents a generalized Mittag-Leffler function, and  $c$  is a random variable having a gamma type density

$$g(c) = \frac{\omega^\mu}{\Gamma(\mu)} c^{\mu-1} e^{-\omega c}, \quad \omega, \mu > 0, \quad 0 < c < \infty.$$

Hence the unconditional density will be

$$(46) \quad N(t) = \frac{N_0}{\Gamma(\mu)} t^{\mu-1} [1 + b(\alpha - 1)t^\nu]^{-\frac{1}{\alpha-1}},$$

with  $\gamma + 1 = 1/(\alpha - 1)$ ,  $\alpha > 1 \rightarrow \gamma = (\alpha - 2)/(\alpha - 1)$  and  $\omega^{-\nu} = b(\alpha - 1)$ ,  $b > 0$ , which corresponds to Tsallis statistics for  $\mu = \nu = b = 1$  and  $\alpha = q > 1$ , physically meaning that the common exponential behavior is replaced by a power-law behavior, including Lévy statistics. Both the translation of the standard reaction equation (43) to a fractional reaction equation (44) and the probabilistic interpretation of such equations lead to deviations from the exponential behavior to power law behavior expressed in terms of Mittag-Leffler functions (45) or, as can be shown for equation (46), to power law behavior in terms of H-functions ( [0] , [0]).

## 5. Conclusion.

In this paper the essentials of pathway idea of Mathai [0] according to different approaches that can be useful for our applications in the theory of time series modelling, fractional calculus and statistical mechanics are established. It is shown that through a parameter  $\alpha$ , called the pathway parameter, one can connect generalized type-1 beta family of densities, generalized type-2 beta family of densities, and generalized gamma family of densities, in the scalar as well as in the matrix cases, also in the real and complex domains. It is shown that when the model is applied to physical situations then the current popular topics of Tsallis statistics and superstatistics in statistical mechanics become special cases of the pathway model, and the model is capable of capturing many stable situations as well as the unstable or chaotic neighbourhoods of the stable situations and transitional stages.

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