Slowing-down of Neutrons: A Fractional Model

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Abstract

The fractional version for the diffusion of neutrons in a material medium is studied. The concept of fractional derivative is presented, in the Caputo and Riesz senses. Using this concept, we discuss a fractional partial differential equation associated with the slowing-down of neutrons, whose analytical solution is presented in terms of Fox’s H-function. As a convenient limiting case, the classical solution is recovered.

Keywords: fractional derivative, diffusion of neutrons.

AMS subject classification: 26A33, 35R11, 60G22.

1. Introduction

The so-called calculus of non-integer order, popularly known as Fractional Calculus (FC), is almost as old as the calculus of integer order of Newton and Leibniz. However, in contrast with the calculus of integer order, FC has been granted a specific area of mathematics only in 1974, after the first international congress dedicated exclusively to it [1,2]. Before this congress there had been just sporadic independent papers and books, without a consolidated line, as can be seen in the classical book by Miller & Ross [3].

During the 1980s FC attracted researchers and explicit applications
began to appear in several fields of knowledge, as those collected in the book by Kilbas et al. [4]. After the decade of 1990 FC was completely consolidated and some specific journals and several textbooks were being published. This fact provided a great visibility to the subject and it gained prestige around the world. A first timeline running from 1695 to 1900 was presented by Ross [5]. Another timeline can be seen in [3] and a recent and definitive timeline from 1645 to 2010 is presented in [6–8].

An important advantage of FC is the non-locality associated with the fractional differential equation that describes a particular phenomenon. In this sense, FC is, in some situations, more realistic and this is a reason why FC has become more and more popular. Nowadays, FC can be considered a frontier area in Mathematics and Physics.

Some recent papers involving ordinary and partial differential equations in different fields and in which FC plays an important role are mentioned below. From the mathematical point of view, Soubhia et al. present, discuss and prove new sum rules for the three-parameters Mittag-Leffler function [9]. Contharteze and Capelas de Oliveira present and prove the so-called fundamental theorem of FC associated with the different derivatives mentioned above, namely the Riemann-Liouville derivative, the Caputo derivative, the Weyl derivative and the Riesz derivative [10] while in [11] the author discusses some problems involving special functions with possible generalizations, namely, the Fox’s $H$-function. On the other hand, from the physical point of view, we may mention the book by Mainardi [12], who discusses anomalous diffusion and waves in linear viscoelasticity; Capelas de Oliveira et al. [13] present a discussion of relaxation in dielectrics; Figueiredo et al. [14] discuss the so-called telegraph equation and obtain a solution given in terms of Fox’s $H$-function; Costa and Capelas de Oliveira [15] solve the fractional wave-diffusion equation with periodic conditions and recover some recent results. More recent are the study of the so-called fractional Schrödinger equation with a delta potential in [16] and the tunneling problem studied in [17].

We concentrate our attention on anomalous diffusion, a topic that plays important roles in many problems involving fractional partial differential equations. Here, we present and solve the so-called fractional slowing-down of neutrons, a particular problem associated with diffusion in a material. The classical version of this problem was studied by Sneddon [18]. To deal with the corresponding fractional problem we adopt the integral transform methodology, introducing the Caputo derivative in the time variable and using the Riesz derivative for the space variable. With a convenient limiting process we recover the classical result. In passing, we present a brief discussion about the problems at the origin in the calculation of the Laplace
transform [19].

The paper is organized as follows: Section 2 reviews the derivatives in the Caputo and Riesz sense; Section 3 brings a brief review of the classical treatment of the problem of slowing-down of neutrons; Section 4 discusses the problems appearing in calculating the Laplace transform at the origin, a topic important to discuss in all problems associated with the methodology of Laplace transform. Our main result appears in Section 5, where we present the so-called fractional slowing-down of neutrons, a problem whose analytical solution is written in terms of the Fox’s $H$-function, and we plot some graphics associated with the main problem for which the classical solution appears as a limiting case. Some convenient relations involving Fox’s $H$-function are presented in Appendix. Concluding remarks close the paper.

2. The Caputo and Riesz derivatives

In this section we introduce the derivatives in the form proposed by Caputo [20], which is more restrictive than the Riemann-Liouville, and the derivatives as proposed by Riesz [21], a definition suited to solve fractional differential equations with boundary conditions by means of Fourier transforms.

2.1. The Caputo derivatives

Let \((I_0^\nu \phi)(t)\) be the so-called fractional integral of order \(\nu\), that is,

\[
(I_0^\nu \phi)(t) = \frac{1}{\Gamma(\nu)} \int_0^t \frac{\phi(\tau)}{(t-\tau)^{1-\nu}} d\tau = \phi(t)^* \frac{t^{\nu-1}}{\Gamma(\nu)},
\]

with \(\text{Re} \nu > 0\). Let \(n\) be a positive integer and \(\mu \in \mathbb{C}\) such that \(\text{Re}(\mu) > 0\). We introduce the fractional derivative of order \(\mu\) in the Caputo sense, \(C\mathcal{D}_t^\mu\), by means of

\[
C\mathcal{D}_t^\mu f(t) = \begin{cases} 
I_{0+}^{n-\mu} f^{(n)}(t), & n - 1 < \text{Re}(\mu) < n \\
 f^{(n)}(t) = D_t^n f(t), & \mu = n.
\end{cases}
\]

The Caputo derivative has been used by many authors in several physical applications. The reason for this choice is the fact that the initial conditions associated with the fractional partial differential equation are usually expressed in terms of integer order derivatives. Let us see and discuss this in connection with the Riemann-Liouville definition of fractional derivative. The Riemann-Liouville derivative is defined as

\[
\mathcal{R}\mathcal{L}\mathcal{D}_t^\mu f(t) = \begin{cases} 
D_t^n I_{0+}^{n-\mu} f(t), & n - 1 < \text{Re}(\mu) < n \\
 f^{(n)}(t) = D_t^n f(t), & \mu = n.
\end{cases}
\]
Let us consider the Laplace transform $\mathcal{L}$, usually defined as

$$\mathcal{L}[f(t)](s) = F(s) = \int_0^\infty e^{-st} f(t) \, dt.$$  

Since the fractional integral is given by a convolution, its Laplace transform is,

$$\mathcal{L}[I_0^\alpha \phi(x)](s) = s^{-\alpha} \mathcal{L}[\phi(x)](s),$$

where $s$ is the parameter of the Laplace transform. Now, from the definition of Caputo derivative, we have that

$$\mathcal{L}[C_D^\mu f(t)](s) = s^{\mu-n} \mathcal{L}[f^{(n)}(t)](s) - \sum_{k=0}^{n-1} s^{\mu-k-1} f^{(k)}(0),$$

while for the Riemann-Liouville definition we have

$$\mathcal{L}[RL_D^\mu f(t)](s) = s^n \mathcal{L}[I_0^{n-\mu} f(t)](s) - \sum_{k=0}^{n-1} s^{k} (I_0^{n-\mu} f)^{(n-k-1)}(0)$$

$$= s^{\mu} \mathcal{L}[f(t)](s) - \sum_{k=0}^{n-1} s^{k} RL_D^{\mu-k-1} f(0).$$

We see therefore that the solution of initial value problems by the Laplace transform technique requires the knowledge of $f^{(k)}(0)$ in the case of the Caputo derivative and of $RL_D^{\mu-k-1} f(0)$ in the case of Riemann-Liouville derivative. Since we will use in the sequel the Laplace transform, we choose to use in this paper the fractional derivative in the Caputo sense. However, this problem – the initialization problem – is much more subtle, and for a detailed discussion we suggest [22].

Eq. (1) and Eq. (4) are satisfactory as long as we are considering continuous functions or piecewise continuous functions with discontinuities not located at the end points of the integration interval. In the case, for example, of causal functions or impulse functions centred at the initial extreme, it is necessary to slightly modify them. As discussed, for example, in [12] (page 10) and [19], the lower bound of integration in Eq. (1) and Eq. (4) has to be replaced by its left-sided limit, that is,

$$\int_{0^-} = \lim_{\epsilon \to 0^-} \int_\epsilon = \lim_{\epsilon \to 0} \int_\epsilon.$$
As a consequence, in Eq.(7) and Eq.(9) the quantities $f^{(k)}(0)$ have to be replaced also by their left-sided limits, that is,

$$f^{(k)}(0^-) = \lim_{\epsilon \to 0^-} f^{(k)}(\epsilon) = \lim_{\epsilon \to 0^+} f^{(k)}(\epsilon).$$

2.2. The Riesz derivatives

The Riesz derivative appears in fractional diffusion process that models anomalous diffusion, in particular in the continuous time random walk (CTRW) approach by an appropriated limit involving the waiting times and the jump widths [23].

We define the fractional derivative of order $\alpha$ in the Riesz sense as

$$R^\alpha_D f(x) = -F^{-1} [|p|^\alpha \mathfrak{F} [f(x); p]; x],$$

where $1 < \alpha \leq 2$ and $p$ is the parameter of the Fourier transform $\mathfrak{F}$. This can be generalized for $n$ spatial dimensions as a fractional laplacian, that is,

$$-(\nabla_x)^\alpha f(x) = -F^{-1} [|p|^\alpha \mathfrak{F} [f(x); p]; x],$$

where $x$ and $p$ are now $n$-dimensional conjugate variables.

3. Slowing-down of neutrons

The study of neutron transport is important, for example, in problems involving nuclear reaction [24]. The transport equation describing the slowing down of neutrons in material media can be reduced, under some assumptions (age approximation), to the age-diffusion equation, which can be written as [18].

$$\frac{\partial}{\partial t} u(x, t) = \nabla^2 u(x, t) + T(x, t),$$

where $u(x, t)$, called slowing-down density, represents the number of neutrons per unit volume, $t$ and $x$ are respectively the time variable (age) and space variables $x = (x_1, \ldots, x_n)$ and $T(x, t)$ is a source function.

Let us consider the case of the spatial dimensional equal to one and with the source term given by $T(x, t) = \delta(x)\delta(t)$. We suppose an infinite medium with slow neutrons, which can be represented by means of the following partial differential equation and initial and boundary conditions:

$$\left\{ \begin{array}{l}
\frac{\partial}{\partial t} u(x, t) = \frac{\partial^2}{\partial x^2} u(x, t) + \delta(x)\delta(t) \\
 u(x, 0) = \delta(x) \\
 \lim_{|x| \to \infty} u_t(x, t) = 0 
\end{array} \right.$$
whose solution is

\[ u(x, t) = \frac{1}{2\sqrt{\pi t}} e^{-x^2/4t} \theta(t), \]

where \( \theta(t) \) is the Heaviside step function. The solution in [18] was obtained by means of a Fourier transform in the space variable and by using an integrating factor in the time variable. The same solution must be obtained, for example, using the Laplace transform in the time variable; however, we need to take some care in this case.

4. The Laplace transform: troubles at the origin

Let us define the Fourier transform of \( u(x, t) \) as

\[ \mathcal{F}[u(x, t)] \equiv U(p, t) = \int_{-\infty}^{+\infty} u(x, t) e^{ipx} \, dx. \]

Taking the Fourier transform in both sides of Eq.(13) and using the boundary conditions we obtain the problem, i.e.,

\[
\begin{align*}
\frac{d}{dt} U(p, t) &= -p^2 U(p, t) + \delta(t), \\
U(p, 0) &= 1.
\end{align*}
\]

However, the presence of \( \delta(t) \) in the above differential equation implies that \( U(p, t) \) must be discontinuous at \( t = 0 \), that is, we must have

\[ \lim_{t \to 0^+} U(p, t) - \lim_{t \to 0^-} U(p, t) = 1. \]

What is, therefore, the meaning of the condition \( U(p, 0) = 1 \)? Since the problem is defined for \( t \geq 0 \), the causality principle implies that we must have \( U(p, t) = 0 \) for \( t < 0 \), and therefore

\[ \lim_{t \to 0^+} U(p, t) = U(p, 0^+) = 1. \]

Now multiplying Eq.(15) by the integrating factor \( e^{p^2 t} \) and integrating it we obtain

\[ U(p, t) = e^{-p^2 t}[\theta(t) + C], \]

where \( C \) is an arbitrary constant and \( \theta(t) \) is the Heaviside step function. Considering the initial condition, we obtain the solution

\[ U(p, t) = \theta(t) e^{-p^2 t}. \]
On the other hand, let us solve Eq.(15) by means of the Laplace transform. Introducing the Laplace transform with parameter $s$, Eq.(15) is converted into an algebraic equation,

$$s \hat{U}(p, s) - U(p, 0) = -p^2 \hat{U}(p, s) + 1,$$

whose solution is given by

$$\hat{U}(p, s) = \frac{U(p, 0) + 1}{s + p^2}.$$

In [19] we can found a detailed discussion concerning the lower limit of integration in the definition of the Laplace transform. This question reflects on the above equation in the interpretation of the term $U(p, 0)$ as $U(p, 0^+)$ or as $U(p, 0^-)$. The conclusion in that we must interpret this as $U(p, 0^-)$, that is, the Laplace transform with $0^-$ in the lower limit of integration,

$$\mathcal{L}[f(t)] = \lim_{\epsilon \to 0^-} \int_{\epsilon}^{\infty} e^{-st} f(t) \, dt = \int_{0^-}^{\infty} e^{-st} f(t) \, dt,$$

and the property of the Laplace transform of a derivative is interpreted as

$$\mathcal{L}[f'(t)] = s \mathcal{L}[f(t)] - f(0^-).$$

Interpreting $U(p, 0)$ in the above equation as $U(p, 0^-)$, which in the present case is

$$(19) \quad U(p, 0^-) = 0,$$

we obtain

$$\hat{U}(p, s) = \frac{1}{s + p^2},$$

whose inversion gives the solution as given in Eq.(18).

5. Fractional slowing-down of neutrons

In this section, which constitutes our main result, we consider, from a phenomenological perspective, a fractional partial differential equation in which the Caputo fractional derivative is associated with the time variable and the Riesz fractional derivative is associated with the space variable. We then consider the problem associated with Eq.(12) in a fractional version in which, using these derivative operators, we include the effect of long memory in the classical slowing-down of neutrons, i.e., we evaluate the analytical solution of the so-called fractional slowing-down of neutrons.
Let us starting recalling Eq.(13), rewritten in the form

\begin{align}
\begin{cases}
\frac{\partial}{\partial t} u(x, t) = \frac{\partial^2}{\partial x^2} u(x, t) + F(x) \delta(t) \\
u(x, 0^-) = 0 \\
\lim_{|x| \to \infty} u_t(x, t) = 0
\end{cases}
\end{align}

(20)

where the source term is \( T(x, t) = F(x) \delta(t) \) and from the discussion of the previous section we replaced the initial condition \( u(x, 0^+) = F(x) \) by \( u(x, 0^-) = 0 \). Then, consider the following fractional partial differential equation with its initial and boundary conditions:

\begin{align}
\begin{cases}
_{C}D_{t}^{\alpha} u(x, t) = -(-\nabla)^{\beta/2} u(x, t) + F(x) \Delta_{a}(t), \\
u(x, 0^-) = 0, \\
\lim_{|x| \to \infty} u_t(x, t) = 0,
\end{cases}
\end{align}

(21)

where \( 0 < \alpha \leq 1 \) and \( 1 < \beta \leq 2 \), \( _{C}D_{t}^{\alpha} \) is the Caputo fractional derivative of order \( \alpha \), and \( (-\nabla)^{\beta/2} \) is the Riesz fractional derivative of order \( \beta \), both introduced in Section 2, and where we have defined

\begin{align}
\Delta_{a}(t) = \frac{t^{-a}}{\Gamma(1-a)} H(t),
\end{align}

(22)

where \( 0 < a \leq 1 \), which is sometimes called Gelfand-Shilov distribution, and is such that

\begin{align}
\lim_{a \to 1} \Delta_{a}(t) = \delta(t).
\end{align}

To solve this problem, we apply the Fourier transform in the space variable and the Laplace transform in the time variable. Applying the Fourier transform in Eq.(21) and using the relation given by Eq.(10) we get

\begin{align}
_{C}D_{t}^{\alpha} U(p, t) = -|p|^{\beta} U(p, t) + \psi(p) \Delta_{a}(t),
\end{align}

(23)

with \( U(p, t) = \mathcal{F}[u(x, t)] \) and \( \psi(p) = \mathcal{F}[F(x)] \).

We now apply the Laplace transform in Eq.(23). As we have discussed in the last section, the lower limit of integration in the Laplace transform is \( 0^- \), and if we remember Eq.(5), Eq.(6) and Eq.(7), we see that for the Caputo derivative

\begin{align}
\mathcal{L}[_{C}D_{t}^{\alpha} U(p, t)](s) = s^{\alpha} \mathcal{L}[U(p, t)] - s^{\alpha-1} U(p, 0^-).
\end{align}

(24)
Since we suppose, like in the usual case, that \( u(x, 0^-) = 0 \), we obtain, after applying the Laplace transform to Eq.(23) that
\[
\hat{U}(p, s) = \psi(p) \frac{s^\alpha-1}{s^\alpha + |p|^\beta}.
\]

Before performing the inverse transforms, let us discuss this problem using the initial value theorem of the Laplace transform, which in this case reads
\[
\lim_{t \to 0^+} U(p, t) = \lim_{s \to \infty} s \hat{U}(p, s).
\]

Therefore, for \( u(x, t) = \mathcal{F}^{-1}[U(p, t)] \), we see that
\[
\lim_{t \to 0^+} u(x, t) = \begin{cases} 
0, & a < \alpha, \\
F(x), & a = \alpha, \\
\infty, & a > \alpha.
\end{cases}
\]

Therefore, we choose
\[
a = \alpha
\]
in order to have the condition
\[
\lim_{t \to 0^+} u(x, t) = F(x)
\]
satisfied. Therefore, if we had chosen \( a = 1 \) – in such a way that the source term in the RHS of Eq.(21) be \( F(x)\delta(t) \) – then the initial condition would not be satisfied (unless \( a = 1 \)).

Now we proceed with the corresponding inverse transforms. We first evaluate the inverse Laplace transform. To this end we use the relation [25]
\[
\mathcal{L}^{-1} \left\{ \frac{s^\alpha-\gamma}{s^\alpha \pm B} \right\} = t^{\gamma-1} E_{\alpha,\gamma}(\mp Bt^\alpha) \theta(t),
\]
where \( B \) is a real constant and \( E_{\alpha,\gamma} (\cdot) \) is the so-called two-parameter Mittag-Leffler function [26], with \( \alpha > 0 \) and \( \gamma > 0 \). Thus we can write
\[
U(p, t) = \psi(p) E_{\alpha}(-|p|^\beta t^\alpha) \theta(t)
\]
with \( 0 < \alpha \leq 1, 1 < \beta \leq 2 \), which is the solution of Eq.(23), and where \( E_{\alpha} (\cdot) = E_{\alpha,1} (\cdot) \) is the one-parameter Mittag-Leffler function.
To proceed with the corresponding inverse Fourier transform we introduce the following function:

$$E^{c}_{a,b}(t,y,\gamma) = t^{b-1}E^{c}_{a,b}(-K|y|^\gamma t^\alpha),$$

with $K$ a real constant, $y = (y_1,\ldots,y_n)$ and $a,b,\gamma > 0$. This function has the following property [14], associated with the Fourier transform:

$$\mathcal{F}^{-1}[E^{c}_{a,b}(t,y,\gamma)] = t^{b-1} \left[ \frac{|x|^\gamma}{2\gamma Kt^\alpha} \binom{1}{1}, (b,\alpha) \right]$$

$$\left. H_{2,3}^{2,1}(z|:) \right|_{c,1}, (n/2,\gamma/2), (1,\gamma/2),$$

with $x = (x_1,\ldots,x_n)$ and $H_{2,3}^{2,1}(z|:) \text{ is the so-called Fox’s } H\text{-function explained in Appendix.}$

Finally, using Eq.(31) we obtain the inverse Fourier transform of Eq.(29), which can be written as follows:

$$u(x,t) = \int_{\mathbb{R}^n} F(x-y)\Phi(y,t) \, dy,$$

where

$$\Phi(x,t) = \frac{\theta(t)}{\sqrt{\pi|x|^n}} H_{2,3}^{2,1} \left[ \frac{|x|^\beta}{2^\beta t^\alpha} \binom{1}{1}, (b,\alpha) \right]$$

$$\left. H_{2,3}^{2,1}(z|:) \right|_{c,1}, (n/2,\beta/2), (1,\beta/2),$$

which is the analytic solution of the problem given by Eq.(21), i.e., the fractional partial differential equation associated with the fractional slowing-down of neutrons.

As an example, let us consider the case of $F(x)$ given by

$$F(x) = \begin{cases} 1, & |x| < 1/2, \\ 0, & |x| > 1/2. \end{cases}$$

For $n = 1$ and $\beta = 2$, the Figure 1 gives the plot of $u(x,t)$ for different values of $\alpha$ and $t$.

When $F(x) = \delta(x)$, we obtain

$$u(x,t) = \frac{\theta(t)}{\sqrt{\pi|x|^n}} H_{2,3}^{2,1} \left[ \frac{|x|^\beta}{2^\beta t^\alpha} \binom{1}{1}, (b,\alpha) \right]$$

$$\left. H_{2,3}^{2,1}(z|:) \right|_{c,1}, (n/2,\beta/2), (1,\beta/2),$$

This mathematical expression is remarkable, after all it has the same functional expression for different values of the order of time and space derivatives and for the dimension $n$ of the space variable. This a very interesting consequence of the use of Fox’s $H\text{-function.}$

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Let us compare our solution with the “non-fractional” case. We consider the particular limiting case, $\alpha \to 1$ and $\beta \to 2$, with $n = 1$, in order to recover the classical solution obtained in Sneddon’s book [18]. Introducing these values in Eq.(35) we can write

$$u(x, t) = \theta(t) \frac{1}{\sqrt{\pi |x|}} H_{2,3}^{2,1} \left[ \frac{|x|^2}{2^2 t} \left( 1, 1 \right), (1, 1), \left( 1/2, 1 \right), (1, 1) \right].$$

Using the relation given by Eq.(42) with $\alpha = 1/2$ and $x = |x|/2\sqrt{t}$, Eq.(36) can be rewritten in the following form:

$$u(x, t) = \frac{\theta(t)}{2 \sqrt{\pi t}} H_{2,3}^{2,1} \left[ \frac{|x|^2}{2^2 t} \left( 1/2, 1 \right), (1/2, 1), \left( 1/2, 1 \right), (0, 1) \right].$$

On the other hand, the definition of the Fox’s $H$-function, Eq.(40),
permits us to write the integral representation
\[
\begin{align*}
    u(x,t) &= \frac{\theta(t)}{2\sqrt{\pi t}} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{\Gamma(\frac{1}{2} + s)\Gamma(s)\Gamma(\frac{1}{2} - s)}{\Gamma(\frac{1}{2} - s)\Gamma(\frac{1}{2} + s)} \left(\frac{x^2}{4t}\right)^{-s} ds \\
    &= \frac{\theta(t)}{2\sqrt{\pi t}} \int_{-i\infty}^{+i\infty} \Gamma(s) \left(\frac{x^2}{4t}\right)^{-s} ds \\
    &= \frac{\theta(t)}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}}.
\end{align*}
\]
which is the classical Sneddon solution given by Eq. (14).

6. Concluding remarks

A particular case of an anomalous diffusion problem, involving the slowing-down of neutrons, was discussed. The discussion was carried in terms of fractional derivatives in the Riesz and Caputo senses. In order to solve the fractional partial differential equation the juxtaposition of Laplace and Fourier transforms was used and the analytical solution was presented in terms of Fox’s \( H \)-function. The classical result due to Sneddon was recovered with a convenient limiting process.

Acknowledgment

ECG is grateful to CNPq, for the research grants. We are indebted to Dr. J. Emílio Maiorino for some remarks and for useful discussions. We are also grateful to the anonymous referee for suggestions that improved the paper.

A. Fox’s \( H \)-Function

Fox’s \( H \)-function, also known as \( H \)-function or Fox’s function, was introduced in the literature as a Mellin-Barnes type integral [27].

Let \( m, n, p \) and \( q \) be integer numbers. Consider the function
\[
\Lambda(s) = \frac{\prod_{i=1}^{m} \Gamma(b_i - B_i s) \prod_{i=1}^{n} \Gamma(1 - a_i + A_i s)}{\prod_{i=m+1}^{q} \Gamma(1 - b_i + B_i s) \prod_{i=n+1}^{p} \Gamma(a_i - A_i s)},
\]
with \( 0 \leq m \leq q \) and \( 0 \leq n \leq p \). The coefficients \( A_i \) and \( B_i \) are positive real numbers; \( a_i \) and \( b_i \) are complex parameters.
The Fox’s $H$-function, denoted by,
\[ H_{p,q}^{m,n}(x) = H_{p,q}^{m,n}(x \mid (a_p, A_p) (b_q, B_q)) = H_{p,q}^{m,n}(x \mid (a_1, A_1) \cdots (a_p, A_p) (b_1, B_1) \cdots (b_q, B_q)) \]
is defined, for $x \neq 0$, as the inverse Mellin transform of $\Lambda(s)$, i.e.,
\[ (40) H_{p,q}^{m,n}(x) = \frac{1}{2\pi i} \int_{L} \Lambda(s) x^s ds, \]
where $\Lambda(s)$ is given in Eq.(39) and the contour $L$ runs from $L - i\infty$ to $L + i\infty$ separating the poles of $\Gamma(1 - a_i + A_i s)$, $(i = 1, \ldots, n)$ from those of $\Gamma(b_i - B_i s)$, $(i = 1, \ldots, m)$. The complex parameters $a_i$ and $b_i$ are determined by imposing that no poles in the integrand coincide. In this Appendix we mention only three properties associated with the Fox’s $H$-function.

P.1. Change of independent variable

Let $c$ be a positive constant. We have
\[ (41) H_{p,q}^{m,n}(x \mid (a_p, A_p) (b_q, B_q)) = c H_{p,q}^{m,n}(x^c \mid (a_p, cA_p) (b_q, cB_q)). \]
To demonstrate this expression it is enough to introduce a change of variable $s \rightarrow cs$ in the integral of the inverse Mellin transform.

P.2. Change of the first argument

Let $\alpha \in \mathbb{R}$. Then we have
\[ (42) x^\alpha H_{p,q}^{m,n}(x \mid (a_p, A_p) (b_q, B_q)) = H_{p,q}^{m,n}(x \mid (a_p + \alpha A_p, A_p) (b_q + \alpha B_q, B_q)). \]
To prove this expression we first introduce the change $a_p \rightarrow a_p + \alpha A_p$ and then take $s \rightarrow s - \alpha$ in the integral of the inverse Mellin transform.

P.3. Lowering of Order

If the first factor $(a_1, A_1)$ is equal to the last one, $(b_q, B_q)$, we have
\[ H_{p,q}^{m,n}(x \mid (a_1, A_1) \cdots (a_p, A_p) (b_1, B_1) \cdots (b_{q-1}, B_{q-1})(a_1, A_1)) = \]
\[ = H_{p,q}^{m,n-1}(x \mid (a_2, A_2) \cdots (a_p, A_p) (b_1, B_1) \cdots (b_{q-1}, B_{q-1})). \]
To prove this identity it is sufficient to simplify the common arguments appearing in the Mellin-Barnes integral. For more about Fox’s $H$-function, see [28].
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