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# The distribution of prices in Financial Markets: a chaotic approach

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## Abstract

We develop a model to study the distribution of prices in Financial markets. Our approach, introducing a term containing the rapidity of variation of price, in the discrete time reduces itself to a logistic map which exhibits a chaotic behaviour. Considering a memory term we are able to reproduce the distribution of real prices in German Dax, French Cac40 and English Ftse with a good agreement. Our model outlines that, compared to the distribution of prices, economic time series can be described by means of chaotic series with a memory function. Moreover, our approach is computationally cheap as it needs only a set of three parameters whose meaning is clear.

*Keywords:* Distribution of prices, chaotic behaviour, financial markets.

*AMS Subject Classification:* 62P05, 65P20, 91G70

## 1. Introduction

In 1874 Leon Walras [1] developed the so called tâtonnement to give a simple description of the law of supply and demand, that was able to catch the evolution of a general equilibrum process. The classical concept of competitive market states that, if supply and demand are out of balance, there is an adjustment related to the excess demand.
In 1985 Border [2] gave a formulation of competitive equilibrum applying variational inequalities inserting himself in a general framework that allowed to solve many hard mathematical physical problems by means of variational formulation [3]. Dafermos [4] and Zhao [5] have investigated the existence of mathematical solutions in the static case, recently generalized by M.B. Donato et al. [6] in the dynamical case.
Although it was long well-known, in spite the absence of a formal proof, that tâtonnement process might not lead to equilibrium, only in the last

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decades it was recognized that chaotic price behavior can be generated within a tâtonnement process. Many authors contributed to this result, as Saary [7] and Weddepohl [8].
In our paper, embracing this framework, we develop a model arguing a relation between supply and demand in which it is also taken into account the rapidity of variation of price. Our approach, in the discrete time, reduces to a logistic map that has a chaotic behavior.
There are two opposite points of view dealing with financial markets. The first one is the Fundamental Analysis in which the intrinsic value of every share is based on supply and demand in which all the relevant variables are contained. The second one, called Technical Analysis, is based on the assumption that history repeats itself and therefore it considers the past share values. The relevance of memory term can be also explained in terms of market frictions that create a time lag in the formation of agent expectation. This is related to the fact [9] that agents do not form their estimations only by a mechanism of rational expectations, but also other features influence the state of market: the trend, investor sentiment and so on.
In our conception we create a link between these two opposite interpretations. In fact we argue that there is an *effective* price deriving from the sum of two different contributions: the former is the solution of the chaotic model proposed to describe the competitive equilibrium and the latter takes into account the past price values derived within the model. The effect of the history of price is taken into account by means of a memory term in which two different parameters are relevant: the duration of memory and the strength of memory.
The goodness of our hypothesis is corroborated by the fact that we are able to reproduce, with a good agreement, the distribution of real prices in German Dax, French Cac40 and English Ftse.
In the last decades a great deal of efforts have been devoted to describe the financial markets in order to obtain a full description of empirical observations and to try to make forecasts. Performing predictions in financial markets has always been the golden goal because of its evident practical significance; and so it has drawn the attention of many researches: it would represent the possibility of nullifying the risk (or at least to reduce it in a significant way) in the process of investment management.
In 1914 Bachelier [10] seamed to kill this dream proposing the theory of random walk to characterise the changes of security prices through time. Fama [11] showed that empirical evidence confirmed the random walk hypothesis: a series of price changes has no memory. This led to the efficient market hypothesis (EMH), in which the basic idea is the following: if a significant statistically correlation exists, it creates an asimmetry that is

filling by community of financial analysts. As a consequence, they will immediately create a new simmetry without possibility of forecasts.
This line of thought has always been received with a lot of scepticism by the professional community devoted to the use of charts and technical analysis rules. Professionals have always claimed that classical statistical tests are mainly linear and therefore, unable to capture the complex patterns that price changes exhibit.
In this last years, by means of deeper investigations that have showed the complex behavior of prices, many authors have accepted that EMH is not completed valid and have developed techniques to "break" the random walk hypothesis. Several methods have been developed as: local and global linear method [12], chaotic attractors [13], a wavelet approach [14], [15].
Our approach could also give a contribution in this framework as it is based on a chaotic time series. Chaos does not allow long term predictions but suggests possibilities for short-time predictions. So, our work could also open new interesting perspectives in the possibility to make forecasts.

## 2. Model

In literature many contributes have been devoted to to find the dynamic price equilibrium solution in pure exchange markets when many agents act within a competitive equilibrium and the use of demand function deriving in Cobb Douglas utility framework is massive (for a valid description of concerns and features see [16], [17], [18]).
Let us consider a financial market with buyers and sellers that insert their pledges exchanging stocks. We imagine that there are N agents (with index i) and M shares (with index j and a price $p_j(t)$).
We can define the VWAP (volume-weighted average price) $p(t)$ considering the weighted sum of the different shares:

$$p(t) = \frac{\sum_j V_j(t)p_j(t)}{\sum_j V_j(t)} \tag{1}$$

where $V_j(t)$ represents the volume of the exchanges associated with the $j^{th}$share dealing with the instant $t$.
With the previous position, we have simplified the problem in such a way that there are N agents interacting and a single share with the price $p(t)$.
It happens in real financial markets when investors trade with Exchange Traded Funds (ETFs), funds whose assets consist of a basket of stocks deposited by institutional investors. In other words ETFs track an index, but can be traded like a stock. These funds are having a greater and greater success because of ETFs are very feasible instruments: they are more tax-

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efficient than normal mutual funds, and, since they track indexes, operating and transaction costs are very low.
The price can be always renormalized in a given interval and so we can imagine that price belongs to the interval [0,1]:

$$0 \leq p(t) \leq 1 \tag{2}$$

Considering these N agents at the t time we have the following share endowment vector :

$$\mathbf{e}(t) = (e_1(t), e_2(t), \ldots\ldots\ldots\ldots\ldots\ldots, e_N(t)) \tag{3}$$

where each $e_i(t)$ is the the amount of shares in possession of every agent that changes because of the trading exchanges. Similarly we define the share demand vector:

$$\mathbf{x}(t) = (x_1(t), x_2(t), \ldots\ldots\ldots\ldots\ldots\ldots, x_N(t)) \tag{4}$$

It is fundamental the excess demand function as it drives price to equilibrum. Since in our system there is no production but only exchange, the excess demand function $z(p(t))$ [16] is defined as follows:

$$z(p(t)) = \sum_i x_i(t) - \sum_i e_i(t) \tag{5}$$

In particular, in a market in which there are N agents and M shares the agent demand function, deriving from Cobb-Douglas utility function, [6] assumes the following form:

$$x_i^j(t) = \frac{\alpha_i(t) \sum_{j=1}^{M} e_i^j(t) p^j(t)}{p^j(t)} \qquad i = 1, .., N;\ i = 1, .., M. \tag{6}$$

As there is an unique global price, posing M=1 and considering $\alpha_i(t) = 1 + \psi_i(t)$, the previous formula reduces to:

$$x_i(t) - e_i(t) = \psi_i(t) e_i(t) \tag{7}$$

It means that the difference between demand $x_i(t)$ and the corresponding endowment requirement $e_i(t)$ of each agent is directly proportional to $e_i(t)$. Expressing each component of endowment vector $e_i(t)$ in terms of its initial value $e_i(0)$,

$$e_i(t) = f_i(t) e_i(0) \tag{8}$$

and posing,

$$\lambda_i(t) = \psi_i(t) f_i(t) \tag{9}$$

we obtain:

$$x_i(t) = (f_i(t) + \lambda_i(t)) e_i(0) \tag{10}$$

Dealing with the function $\lambda_i(t)$, we imagine that it depends on the time $t$ through both price $p(t)$ and the rapidity of evolution $\frac{dp(t)}{dt}$ and so:

$$\lambda_i(t) = \lambda_i(p(t), \frac{dp(t)}{dt}) \tag{11}$$

Our hypothesis is built on a basic idea: an analogy with a damped harmonic oscillator in which both elastic force and damped frictional one act as split contributions. So we can separate $\lambda_i(t)$ into two terms $\lambda_{i1}(t)$ and $\lambda_{i2}(t)$:

$$\lambda_i(t) = \lambda_{i1}(t) + \lambda_{i2}(t) \tag{12}$$

The former, $\lambda_{i1}(t)$, depends on the price:

$$\lambda_{i1}(t) = \lambda_{i1}(p(t)) \tag{13}$$

while the latter, $\lambda_{i2}(t)$ is a function of the derivative of price respect to the time:

$$\lambda_{i2}(t) = \lambda_{i2}(\frac{dp(t)}{dt}) \tag{14}$$

Let us investigate the functional form of $\lambda_{i1}(p(t))$ . Because of the previous analogy, considering a potential associated to an elastic force, we argue that $\lambda_{i1}(p(t))$ has a parabolic form.
So we define $\lambda_{i1}(p(t))$ as follows:

$$\lambda_{i1}(t) = k_i(p^2(t) - p(t)) = k_i p(t)(p(t) - 1) \tag{15}$$

where $k_i$ is a positive constant depending on the $i^{th}$ agent attitudes.
Focusing our attention on $\lambda_2(dp(t)/dt)$, we imagine that it is directly proportional to $\frac{dp(t)}{dt}$:

$$\lambda_{i2}(dp/dt) = \varphi_i \frac{dp(t)}{dt} \tag{16}$$

with $\varphi_i$ a positive constant dealing with $i^{th}$ agent.
Replacing Eqs. (15) and (16)in Eq. (5) the excess demand function reduces to:

$$z(p(t)) = p(t)(p(t) - 1) \sum_i k_i e_i(0) + \frac{dp(t)}{dt} \sum_i \varphi_i e_i(0) \tag{17}$$

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In order to obtain the price equilibrum $p^\star(t)$, we have to to find the price that nullifies the excess demand function:

$$z(p^\star(t)) = 0 \tag{18}$$

Defining:

$$k = \sum_i k_i e_i(0); \quad \varphi = \sum_i \varphi_i e_i(0); \quad \mu = \frac{k}{\varphi}$$

and replacing in Eq. (17), $p^\star(t)$ is a competitive equilibrum price if it is solution of the following differential equation:

$$\frac{dp^\star(t)}{dt} = \mu p^\star(t)(1 - p^\star(t)) \tag{20}$$

As our aim is describing the evolution of price time series, we discuss about the discrete case corresponding to the previous differential equation.
The advanteges of such a transformation are computational too: difference equation can be easily solved by means of a cheap iteration method, while the corresponding differential equation needs a greater computational wasting time.
Considering a discretization of time, our differential equation reduces to a difference one [19], that is known as the logistic map:

$$p^\star_{n+1} = \mu p^\star_n (1 - p^\star_n) \tag{21}$$

The logistic map, originally introduced by Verhulst in 1845 to describe the evolution of a population in a limited interval, has been widely described (see for example May paper [20]) in literature.
It is a non linear map (with $0 < \mu \leq 4$) in which regions of chaos (which appear for $\mu \geq 3.569$) and windows of periodicity alternate. In periodicity regions the equilibrum price $p^\star_n$ assumes only a limited number of possible values, while it does not happen in chaotic framework. The passage from periodicity to chaos can be referred to a sequence of period doublings that is a common feature in dynamical systems in which chaos appears as discussed by Baker [19].
The $p^\star_n$ price represents a price without memory of past values.
According to the Technical Analysis concerns, we introduce an *effective* price $\tilde{p_n}$ in order to compute the influence of history of prices. This is defined as the sum between the $p^\star_n$ and the whole deal of past competitive equilibrum prices evaluated within the model in the previous instants. Then we introduce a function of memory that takes into account the duration and the force concerning the effects of the past in agents decisions:

$$\tilde{p_n} = p^\star_n + f(p^\star_j, \alpha, \beta) \qquad j < m \tag{22}$$

where $\alpha$ and $\beta$ are positive constants that represent the duration and the intensity of memory respectively.
As the previous equation reveals, only three parameters $(\alpha, \beta, \mu)$ are necessary to reproduce the distribution of prices in real financial markets. The $\mu$ parameter is related to the logistic map, that represents the solution of a competitive equilibrum problem without frictions and time lags (linked with the Fundamental Analysis); viceversa $(\alpha, \beta)$ parameters, related to Technical Analysis, deal with the agent estimation of price within an adaptive scheme on past price values.

## 3. Results and discussion

We analyze different financial markets, with time series formed by the daily closure data, considering different time intervals evaluating in each case the optimal determination of $(\alpha, \beta, \mu)$ parameters. In each market we compare the distribution of real prices with the one of *effective* prices obtained by means of our model.
The calculations are performed considering a function of memory that is a decaying exponential one. Exponential decay, the decrease at a rate proportional to its value, is a feature that appears in many fields to describe the decay of a perturbation. In our description memory can be considered as a sort of perturbation of Technical Analysis elements respect to an "ideal" stock market related to Fundamental Market. In such a way it is quite natural to consider a decay of memory with an exponential form. Of course other functions could be considered: ex post, the agreement of fitting shows the validity of the choice and so exponential decay of the memory seems to be a realistic hypothesis in our approach. So every *effective* price $\tilde{p_n}$ is evaluated as follows:

$$\tilde{p_n} = p_n^\star + \beta \sum_{m=0}^{n-1} e^{-\alpha(n-m)} p_m^\star \tag{23}$$

Moreover we impose a cut off in such a way that only 100 past enclosure days are considered in the memory term and so:

$$\tilde{p_n} = p_n^\star + \beta \sum_{m=k(n)}^{n-1} e^{-\alpha(n-m)} p_m^\star \tag{24}$$

where $k(n) = max(0, n-100)$.
In order to have a simpler comparison with the real price values, we renormalize both *effective* and real prices in the $[0, 1]$ interval by means the

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following transformation:

$$\hat{p_n} = \frac{\tilde{p}_n - p_{min}}{p_{max} - p_{min}} \tag{25}$$

where $p_{min}$ and $p_{max}$ are the minimum and maximum of prices in the time range considered.
To measure the goodness of our model, we have evaluated in each case the mean square deviations (MSD) between *experimental* distribution and reproduced one (by means of with our best set of parameters) defined as follows:

$$MSD = \frac{\sum_{i=0}^{t}(f_i - f_i^R)^2}{t} \tag{26}$$

where $t$ is the number of subinterval taken into account (in our paper we fixed $t = 10$), $f_i$ represents the frequence in our approach in the $i^{th}$ subinterval,while $f_i^R$ is the frequence in the same subinterval measured in real price distribution.
Our fitting process consists in choosing the set of parameters in such a way that MSD assumes its minimum value.
The constraints imposed in the fitting process are: the $(\alpha, \beta)$ parameters are positive constants, while $\mu$ falls inside the $(0, 4]$ range in such a way that logistic map is well defined.
Each set of parameters has been determined considering, as initial condition of *effective* time series, the first value of renormalized real prices.
In each analyzed financial market we have calculated the volatility of real data determined as the mean square root RMS [21]:

$$RMS = \sqrt{\frac{\sum_{i=0}^{N}(\hat{p}_i - < \hat{p} >)^2}{N}} \tag{27}$$

where $N$ is the number of daily closures and $< \tilde{p} >$ is the mean value in the considered time interval.
In Table 1 we report the best fitting set of parameters referred to the most important European Financial Markets: German Dax, French Cac40 and English Ftse. We analyze the real price distribution considering 3000 daily closure values (from 3-04-1997 to 17-02-2009).
Our approach is able to reproduce with a good agreement *experimental* distribution of prices in each analyzed case. In order to visualize the accuracy of reproduction, in Fig.1 we report the hystograms concerning Dax index.
We observe that the set of parameters changes focusing attention on different markets, and so the set is a feature of the market.

Table 1. We report the set of parameters fitting German Dax, French Cac40 and English Ftse considering a range of 12 years (from 3-04-1997 to 16-02-2009). We observe that the set of parameters is different changing the market.

| Market | $\alpha$ | $\beta$ | $\mu$ | $MSD \cdot 10^{-5}$ | volatility |
|---|---|---|---|---|---|
| Dax | 0.125 | 1.00 | 3.6616 | 8 | 0.23 |
| Cac40 | 0.302 | 1.32 | 3.7800 | 9 | 0.23 |
| Ftse | 0.036 | 0.25 | 3.7650 | 14 | 0.24 |

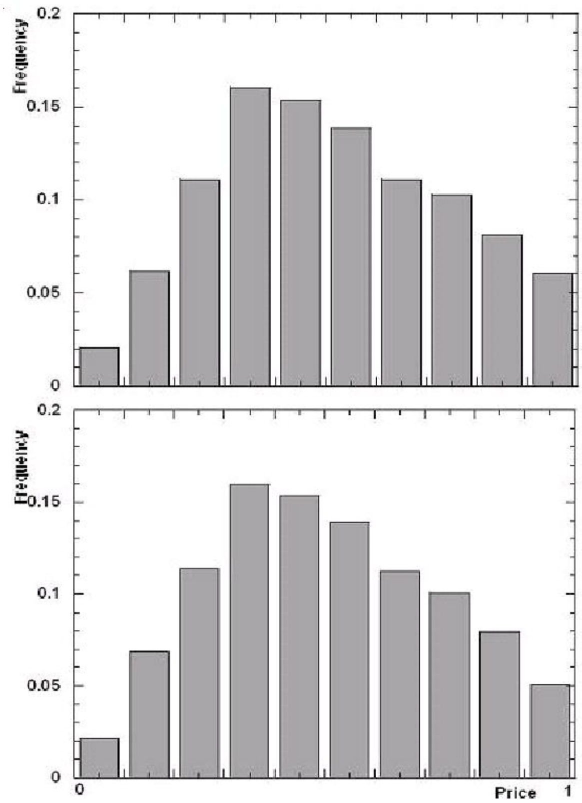


Fig. 1. Distribution of prices considering a period of 12 years (from January 1996 to February 2009) in German Dax index. Upper frame: hystogram with real data. Lower frame: hystogram reproduced in our description with the following set of parameters: $\alpha = 0.125$, $\beta = 1$, $\mu = 3.6616$.

In particular, in each market $\mu$ parameter values fall inside the regions of chaos of logistic map, and so the fitting is driven by a chaotic series.
Within the same market, as Table 2 concerning Dax index shows, the set of best fitting parameters is different varying the investigated interval time. So the set of best fitting parameters depends on both market and temporal

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scale taken into account.
These results are consistent with the Technical Analysis concepts in which the non rational features change varying the market and the time interval.
Both tables reveal that increasing volatility, MSD increases; so the agree-

Table 2. We report the set of parameters fitting German Dax focusing on different time scales. We observe that the set of parameters is different changing the interval. The agreement worsens when volatility increases. 3000, 2000, 1000 (to 16-02-2009) correspond to a starting point of 3-04-1997, 19-03-2001, 4-03-2005 respectively.

| Interval (closure days ) | $\alpha$ | $\beta$ | $\mu$ | $MSD \cdot 10^{-5}$ | volatility |
|---|---|---|---|---|---|
| 3000 | 0.125 | 1.00 | 3.6616 | 8 | 0.23 |
| 2000 | 0.190 | 1.26 | 3.6600 | 13 | 0.24 |
| 1000 | 0.811 | 1.61 | 3.7221 | 19 | 0.28 |

ment between the real price distribution and the *effective* one is worse. This result could seem a tautology, but it is not so obvious in an approach in which the fit is led by a non linear pivot, a chaotic series.
Moreover $\alpha$ and $\beta$ parameters are both in a direct proportionality to volatility. In other words, the increase of volatility creates a reduction of the memory term duration revealed by greater $\alpha$ values. The decrease of $\alpha$ is balanced by the contemporary assumption of greater $\beta$ values, corresponding to an increase of the strength of memory.
So, in presence of high volatility (that is generally, but not always [22], connected with a bubble of speculation), we verify that memory abridges and its intensity increases, but accuracy decreases as MSD reveals in Table 2.
In Table 3 we analyze the parameter values in Dax index during the finan-

Table 3. We report the set of parameters fitting German Dax dealing with the two great crises in 2001 (500 closure days from 16-02-2000) and 2008 (500 closure days from 16-02-2007). Both extreme events are stressed in our description by means the increase of $\beta$ parameter that reveals the strong intensity of memory.

| Interval (closure days ) | $\alpha$ | $\beta$ | $\mu$ | $MSD \cdot 10^{-5}$ | volatility |
|---|---|---|---|---|---|
| 500 | 1.111 | 4.21 | 3.7271 | 18 | 0.29 |
| 500 | 1.107 | 4.27 | 3.7293 | 19 | 0.29 |

cial crises of 2001 and 2008 that, according to the most of economists, has been the the worst one since the Great Depression of the 1930s. These panic and crashes in stock markets around the world, created by this systemic crisis, are stressed, within our model, by an increase of memory strength. In this framework it appears that, within a financial crisis the Technical contribution becomes more relevant, in virtue of the relevance of non ratio-

nal elements.
Further investigations are necessary to improve the agreement in presence of high volatility. Preliminary calculations reveal that the usage of memory functions different from a decaying exponential one, in particular a gaussian function, increase the accuracy of our approach in reproducing real price distributions.
Our description is different from the stochastic approaches used to describe financial properties in stock markets [23], [24], [25]. The most relevant features that distinguish this model respect to the common stochastic volatility ones (GARCH, Heston, jump Merton) are: the limited number of parameters (whose meaning is clear and not esoteric), the use, as fitting pivot, of a chaotic series (respect to the use of random series) and the absence of a pre-definite distribution in performing calculations. These novelties allow to overcame some of the drawbacks that affect other models.
The GARCH [26] approach describes the time evolution of a given economic time series using many control parameters respect to a random variable. Because of the limited number of parameters involved in our approach, we compare it with GARCH(1,1) that has three control parameters (and in general a Gaussian conditional pdf, probability density function). In spite of its massive use, GARCH(1,1) has some serious troubles: first of all, its behavior for different time horizons. For finite variance, the Central Limit Theorem applies and it implies a progressive a decrease in the leptokurtosis of the GARCH(1,1) process. Drost and Nijman [27] showed that a temporal aggregation of GARCH(1,1) continues to be a GARCH(1,1) process with control parameters varying, but the attractor process with finite variance is always the Gaussian process (corresponding to EMH hypothesis). So GARCH(1,1) fails to capture important aspects of financial data: temporal dependence becomes negligible too quickly as the sampling interval increases and so GARCH(1,1) can not sufficiently capture the heavy-tailed nature of many financial series. In other words the GARCH(1,1) memory is not long enough, because the ACF (autocorrelation function) decreases too fast. As a consequence, GARCH(1; 1) does not seem to provide a good description of long series.
This limitation is overcome in our description in which the agreement does not depend on the length of time series (as Table 2 reveals). Moreover GARCH (1, 1) model neglects the so called leverage effect (the market is much more affected by "bad" news compared to "good" news) in stocks as does not distinguish positive and negative shocks and so it is not able to outline stock crashes. Vice versa, in our description these crashes in markets are revealed by an increase of memory strength as Table 3 shows.
In spite of the Heston approach [28] is able to give an accurate reproduction,

by means of a few parameters, the distribution of price returns in large time scales too [29], there is an exoteric element: it contains, as hidden variable, the exogenous stock volatility that can not be directly derived from time series. There are some drawbacks to the approximation of this quantity, including the fact that it neglects potentially useful information. Our model does not contain any hidden variable: the parameters have a clear meaning. The main drawback, however, is that sample paths generated by the Heston conception are continuous and no sudden jumps in the price process can be created. So Heston model is not flexible enough to capture the patterns of smiles and smirks seen in the data. Our approach seems to be able to take into account discontinuities (a relevant contribute in case of stock crashes) because of the chaotic series appears to be more feasible to deal with jumps in stock markets (as Table 3 reveals).
To take into account jumps [30], Merton [31] considers a Brownian motion with drift (continuous diffusion process) plus a compound Poisson process (discontinuous jump process). One limitation of the jump diffusion model is that is the no correlation between volatility in the diffusive component and the strength of jumps: in reality, volatility tends to increase dramatically around the time that large jumps occur. This limit disappears in our description as the jump process can be contained in chaotic series that leads our fitting process and does not appear as a separate contribute respect to the Brownian motion.
The most of improvements of previous models, in order to take into account the tails, introduce regime-switching frameworks that combine two or more sets of model coefficients into one system [32]. In our description this necessity is overcome because of the chaotic price series (the pivot of our fitting process) is able to reproduce the structure of price distributions without adding more and more parameters. This result is consistent with multifractality approaches [33] in which heterogeneity is not considered through time-varying conditional second moments in a discrete time framework, but my means time-varying price path.

## 4. Conclusions

This paper shows a model to reproduce the distribution of prices in financial markets. In our approach the instantaneous price is the sum of two contributions. The first one (linked with the Fundamental Analysis) is derived within a competitive equilibrum model and exhibits a chaotic behaviour, because of it satisfies, in the discrete time, the logistic map equation. The second contribution (taken into account by a memory term of past chaotic prices and so related to Technical Analysis) outlines the

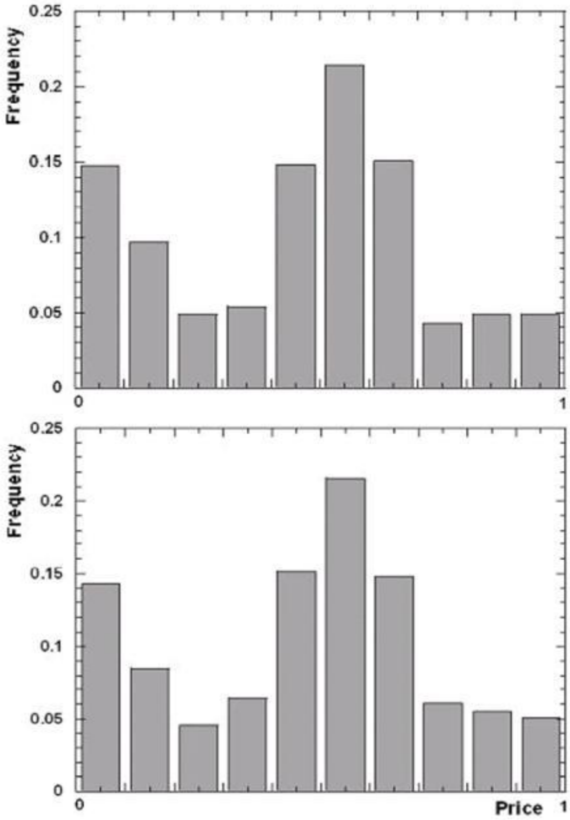


Fig. 2. Distribution of Euro/Dollar exchange rates in the 2200 closure days to 26-02-2009. Upper frame: frequency hystogram obtained reporting real data. Lower frame: distribution with the following values of parameters: $\alpha = 0.674$, $\beta = 3.63$, $\mu = 3.8593$. The square displacement is $MSD = 3 \cdot 10^{-4}$ while the volatility of real prices is $RMS = 0.25$.

effect of the history of prices (the trend, the patterns, the cycles) and other non rational elements (such as the sentiment of agents).
Two features are stressed: first of all, in spite of the limited number of involved parameters, a good description of real distribution data is obtained and then the quality of fit decreases when the volatility increases.
Moreover our model is able to reveal, by means of very great values of the intensity of memory, the great crisis in which we are imbued.
The model is very flexible and it is also able to describe foreign charge rate distributions within a discrete agreement as Fig.2 shows.
Because of our model is based on chaotic series, where short time forecasts are possible, our approach opens new perspectives in developing predictions: the future efforts will be devoted to apply the *effective* prices to make

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statistical forecasts concerning the future values assumed by real prices.

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