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# Optimal control in wastewater management: a multi-objective study

L. J. Alvarez-Vázquez[1], N. García-Chan[2], A. Martínez[1], M. E. Vázquez-Méndez[2]

[1] *Departamento de Matemática Aplicada II, E.T.S.I. Telecomunicación*
*Universidad de Vigo, 36310 Vigo, Spain*
*lino@dma.uvigo.es*
*aurea@dma.uvigo.es*

[2] *Departamento de Matemática Aplicada, E.P.S.*
*Universidad de Santiago de Compostela, 27002 Lugo, Spain*
*netog_g@hotmail.com*
*miguelernesto.vazquez@usc.es*

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## Abstract

This work deals with the management of a sewage treatment system consisting of several purifying plants discharging treated sewage into the same region, for instance, an estuary or a lake. The management of this system involves economical aspects (given by the treatment cost in the plants) and also environmental aspects (related to water quality in the sensitive areas of the region). By assuming a unique manager for all plants, the problem is formulated as a multi-objective parabolic optimal control problem, and it is studied from a cooperative viewpoint. Pareto-optimal solutions are completely characterized, a numerical algorithm to obtain the Pareto-optimal set is proposed and, finally, numerical experiences in the estuary of Vigo (NW Spain) are presented.

*Keywords:* Optimal control, multi-objective, wastewater management.

*AMS Subject Classification:* 49K20, 49M25, 90C29.

## 1. Introduction

Waste elimination is one of the most important environmental problems nowadays. For the particular case of wastewater in urban areas, domestic sewage and trade waste are collected from different districts and transported to purifying plants *via* pipes and pumping stations. These plants treat sewage by different biochemical methods and, finally, treated effluents are discharged through subsurface outfalls into aquatic media (a lake, a river, an estuary...). Sewage treatment is not only a necessary task but also a very expensive one, and determining the intensity of the treatment

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has become a very difficult problem involving both environmental and economical aspects.

By giving priority to environmental aspects, this type of problems has been already studied in the framework of optimal control of partial differential equations (see, for example, [1] for the optimal control of watewater discharges, or [2] for a similar problem related to the control of pollutant emissions to atmosphere). More recently (cf. [3] and [4]) game theory has been introduced in the formulation of the problem by assuming different managers (players) for each plant; however, in this case, only economical objectives are considered (environmental aspects have been treated from an economical point of view).

In this paper we are going to assume a unique manager controlling all plants, but now priorities will not be stated, and economical and environmental aspects will be separate (both of them will be treated as objectives to be minimized). According to this, in section 2 the problem will be described and formulated as a multi-objective parabolic optimal control problem. Section 3 will be devoted to give a complete characterization of the Pareto-optimal solutions. In section 4, we will detail an useful method to compute the objective functionals and their gradients, and propose a numerical algorithm to obtain the Pareto-optimal set. Finally, in section 5, numerical experiences in the estuary of Vigo (NW Spain) will be presented.

## 2. The multi-objective problem

First of all we introduce the environmental problem: we consider an urban area with a wastewater treatment system consisting of $N_E$ purifying plants, which collect sewage from different districts, treat it with different methods and, finally, discharge the treated effluents into a shallow water domain $\Omega$ (for example, an estuary) through submarine outfalls (see a scheme in Fig. 1). Moreover, we assume the existence of several sensitive areas (representing fisheries, beaches or marine recreation zones) where the water quality should be guaranteed with pollution levels lower than allowed thresholds (fixed by administrative directives). The problem consists of determining the intensity of the treatment in each plant along an arbitrary period of time $(0, T)$. The strategy should satisfy two objectives: it should be *inexpensive* (low economic cost), and it should be *green* (low environmental impact).

### 2.1. *Mathematical formulation*

In order to formulate the problem we need to deal with the numerical simulation of the environmental impact caused by pointwise treated

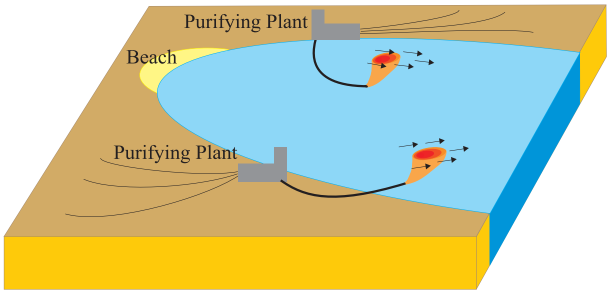


Fig. 1. Scheme of domain $\Omega$

sewage discharges in a shallow water domain. Often (especially in urban areas) environmental impact is assessed in terms of *Faecal Coliform* (FC) concentration. For a time interval $(0,T)$, FC concentration in a shallow water domain $\Omega \subset \mathbb{R}^2$ is given by:

$$(1) \quad \left.\begin{array}{ll} \dfrac{\partial \rho}{\partial t} + \vec{u}.\vec{\nabla}\rho - \beta\Delta\rho + \kappa\rho = \dfrac{1}{h}\displaystyle\sum_{j=1}^{N_E} m_j(t)\delta(x-P_j) & \text{in } \Omega \times (0,T), \\ \rho(0) = \rho_0 & \text{in } \Omega, \\ \dfrac{\partial \rho}{\partial n} = 0 & \text{on } \Gamma \times (0,T), \end{array}\right\}$$

where $\rho(x,t)$ is the depth-averaged FC concentration at point $x = (x_1,x_2) \in \Omega$ and at time $t \in (0,T)$, $h(x,t)$ and $\vec{u}(x,t)$ denote, respectively, the height and the velocity field of water, $\beta > 0$ is a given viscosity coefficient collecting turbulent and dispersion effects, $\kappa \in \mathbb{R}$ is an experimental coefficient related to the loss rate of FC. Sewage discharges are collected into the second member of the partial differential equation: $N_E$ is the number of pointwise discharges, $P_1,\ldots,P_{N_E}$ are the discharge points, $m_j(t)$ is the mass flow rate of FC discharged at $P_j$, and $\delta(x-P_j)$ denotes the *Dirac measure* at $P_j$. Finally, $\rho_0(x)$ is the initial FC concentration, and $\Gamma$ denotes the boundary of $\Omega$.

The problem of determining the sewage treatment intensity in a purifying plant is equivalent to obtaining the mass flow rate of FC discharged after purification. So, the control variables for the problem of the management of the sewage treatment system are functions $m_j(t)$. If we denote by $\overline{m}_j$ the maximum mass flow rate of FC arriving to the $j$-th plant, and by $\underline{m}_j > 0$ the minimum mass flow rate of FC discharging at $P_j$ (corresponding to the maximum purification in that plant), we are looking for functions $m_j$ in the admissible sets:

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$$M_j = \{m \in L^\infty(0,T) \ : \ \underline{m}_j \leq m(t) \leq \overline{m}_j \text{ a.e. in } (0,T)\}, \quad j = 1, \ldots, N_E.$$

Moreover, the cost of the depuration in the $j$-th plant can be assumed to depend on $m_j(t)$ in such a way that a lower level of $m_j(t)$ leads to a more intensive depuration and, consequently, to a higher cost. So, we can assume the existence of cost functions $f_1, \ldots, f_{N_E}$ such that the global economic cost of the treatment system for a time interval $(0,T)$ is given by the functional

$$K_C(m) = \sum_{j=1}^{N_E} \int_0^T f_j(m_j(t))dt, \tag{2}$$

with $m = (m_1, \ldots, m_{N_E})$ the control vector.

On the other hand, we have to take into account a cost related to the environmental impact in sensitive areas. We assume the existence of $N_Z$ areas $A_l \subset \Omega$, $l = 1, \ldots, N_Z$ (representing fisheries, beaches or marine recreation zones), where the FC concentration should be lower than a fixed threshold $\sigma_l$, which depends on the type of area. The environmental impact on $A_l$ can be assessed by the functional

$$K_l(m) = \frac{1}{2\epsilon_l} \int_{A_l \times (0,T)} (\rho(x,t) - \sigma_l)_+^2 dxdt, \tag{3}$$

where $(\rho(x,t)-\sigma_l)_+$ denotes the positive part of $\rho(x,t)-\sigma_l$, that is, $(\rho(x,t)-\sigma_l)_+ = \max\{\rho(x,t) - \sigma_l, 0\}$ and $\epsilon_l$ is a given penalty parameter which depends on the particular characteristics of the zone $A_l$. The value of $\epsilon_l$ should be fixed from ecological preferences and, obviously, it is inversely proportional to the importance given to satisfying FC concentration lower than the threshold level $\sigma_l$.

We also assume the existence of a unique organization (for instance, a government) managing all the plants of the depuration system. Because of it, the economic cost can be treated in a global way (as a unique objective for all plants), but the environmental impact should be divided by areas, because in this manner it can be possible the detection, for a same economical cost, of several depuration options that will favour (from a ecological viewpoint) one area over the other ones. Among the different optimal options, will be the decision maker (manager) who (invoking possible ecological, commercial or political criteria) will select one from them.

In this situation, the management of the wastewater treatment system can be formulated as the following multi-objective parabolic optimal control problem:

**Problem** $(P)$: Find $m(t) = (m_1(t), \ldots, m_{N_E}(t)) \in M = \prod_{j=1}^{N_E} M_j$ which minimizes the economic cost $K_C(m)$, given by (2), and the environmental impacts $K_1(m), \ldots, K_{N_Z}(m)$, given by (3).

2.2. *Mathematical analysis*

From the study of the state system (1) (see [1]) we know that $\rho_{|\cup_{l=1}^{N_Z} \bar{A}_l \times [0,T]} \in C(\cup_{l=1}^{N_Z} \bar{A}_l \times [0,T])$. Moreover, the mapping

$$\begin{array}{rcl} F : (L^\infty(0,T))^{N_E} & \longrightarrow & C(\cup_{i=1}^{N_Z} \bar{A}_i \times [0,T]) \\ m = (m_1, m_2, \ldots, m_{N_E}) & \longrightarrow & F(m) = \rho_{|\cup_{i=1}^{N_Z} \bar{A}_i \times [0,T]} \end{array}$$

is well defined, and it is affine and continuous. Consequently, objective functionals formally introduced by (2) and (3) are well defined on $M$ and can be written as

$$K_C(m) = \sum_{j=1}^{N_E} \Theta_j(m_j), \tag{4}$$

$$K_l(m) = H_l(F(m)),\ l = 1, \ldots, N_Z, \tag{5}$$

where:

- $\Theta_j : M_j \longrightarrow \mathbb{R}$ is given by $\Theta_j(m_j) = \int_0^T f_j(m_j(t))dt$
- $H_l : C(\cup_{k=1}^{N_Z} \bar{A}_k \times [0,T]) \longrightarrow \mathbb{R}$ is defined by $H_l(\rho) = \dfrac{1}{2\epsilon_l} \int_{A_l \times (0,T)} (\rho(x,t) - \sigma_l)_+^2 dxdt$

Regularity of $\Theta_j$ and, consequently, regularity of $K_C$ depends on the regularity of $f_j$. It is obvious that, if $f_j \in C([\underline{m}_j, \overline{m}_j])$ is strictly convex, then $\Theta_j$ also is. Moreover, if $f_j \in C^1([\underline{m}_j, \overline{m}_j])$, then (cf. [1]) $\Theta_j$ is Gateaux differentiable at every point $m_j \in M_j$ and

$$\langle \Theta_j'(m_j), \delta m_j \rangle = \int_0^T f_j'(m_j(t))\, \delta m_j(t)\, dt, \tag{6}$$

for all $\delta m_j \in L^\infty(0,T)$ satisfying $m_j + \epsilon_0 \delta m_j \in M_j$ for some $\epsilon_0 > 0$.

On the other hand, due to the fact that function $(\cdot)_+^2$ is convex and differentiable, $H_l$ is convex, continuous and Gateaux differentiable. Moreover, since $F$ is affine and continuous, functionals $K_1, \ldots, K_{N_Z}$ are also continuous, Gateaux differentiable and convex.

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## 3. Pareto-optimal solutions

We must bear in mind that, due to the contradictory character of the economic cost $K_C(m)$ and the environmental impacts $K_l(m)$, there will not exist a *utopic* (or ideal) solution $m \in M$ minimizing simultaneously all functionals of problem $(P)$. So, we are led to look for a more realistic type of optimal solutions.

If preference specifications are provided before the solution process, we can use an *a priori* method to solve the problem. For example, if priority to environmental aspects is given we should look for a *global ecological-optimal solution*, that is a vector $m \in M$ which minimizes the economic cost guaranteeing null environmental impact in all sensitive areas. We can state the following definition:

**Definition 3.1.** A vector $m \in M$ is a global ecological-optimal solution of problem $(P)$ if it is a solution of the following single-objective optimal control problem

$$(7) \qquad \begin{cases} \text{Minimize } K_C(m) \\ \text{suject to } \ m \in M, \\ \qquad\qquad K_l(m) = 0, \quad \forall l = 1, \ldots, N_Z. \end{cases}$$

Because of $\rho_{|\cup_{l=1}^{N_Z} \bar{A}_l \times [0,T]} \in C(\cup_{l=1}^{N_Z} \bar{A}_l \times [0,T])$, constraint $K_l(m) = 0$ is equivalent to $\rho(x,t) \leq \sigma_l$ in $A_l \times (0,T)$, and, if $f_j \in C([\underline{m}_j, \overline{m}_j])$ is strictly convex for all $j = 1, \ldots, N_E$, problem (7) has a unique solution (see [1]). This solution, the global ecological-optimal solution of problem $(P)$, will be denoted by $m_{ecol}$.

In most cases priorities are not known, and we have to study the problem $(P)$ without the decision maker articulating any preferences among the objectives. In this situation, economic cost $K_C(m)$ and environmental impacts $K_l(m)$ should be treated as objective functionals to be minimized, and we should look for *Pareto-optimal solutions*: solutions which are *optimal* in the sense that there are not *globally-better* solutions bringing off a simultaneously decrease of all objective functionals. Once we obtain all the Pareto-optimal solutions (Pareto-optimal set), the decision maker will have to choose among them. We can give a more formal definition of these concepts (see more details, for instance, in [5]):

**Definition 3.2.** We say that $m \in M$ is a Pareto-optimal solution of problem $(P)$ if there does not exist any $m^* \in M$ such that

*a*) $K_C(m^*) \leq K_C(m)$ and $K_l(m^*) \leq K_l(m)$ for all $l \in \{1, \ldots, N_Z\}$

*b*) $K_C(m^*) < K_C(m)$ or $K_l(m^*) < K_l(m)$ for at least one $l \in \{1, \ldots, N_Z\}$

If $m \in M$ is a Pareto-optimal solution, the objective vector $(K_C(m), K_1(m), \ldots, K_{N_Z}(m)) \in \mathbb{R}^{N_Z+1}$ is also known as Pareto-optimal. The set of Pareto-optimal solutions is called *Pareto-optimal set*, and the set of Pareto-optimal objective vectors is called *Pareto-optimal frontier*.

It is worthwhile remarking here that, from its own definition, $m_{ecol}$ is a Pareto-optimal solution of Problem $(P)$. To obtain other Pareto-optimal solutions can be useful to employ a Multi-objective Optimization Evolutionary Algorithm (MOEA), because it deal simultaneously with a set of possible solutions (the so-called population) which allows us to find several members of the Pareto optimal set in a single run of the algorithm (see [6] for more details). However, in this work, we are going to use the classical *weighting method* (extended to Banach spaces) because it gives us a very useful characterization of the Pareto optimal set and it provides (as we can see later) a good method to obtain the complete Pareto-optimal frontier. In order to do it, for each $\lambda = (\lambda_C, \lambda_1, \ldots, \lambda_{N_Z})^t \in \mathbb{R}^{N_Z+1}$ verifying $\lambda_l \geq 0$ for all $l = 1, \ldots, N_Z$, and $\lambda_C + \sum_{l=1}^{N_Z} \lambda_l = 1$, we introduce the following *weighting problem*:

$$(8) \qquad \begin{cases} \text{Minimize } K_\lambda(m) = \lambda_C K_C(m) + \displaystyle\sum_{l=1}^{N_Z} \lambda_l K_l(m) \\ \text{suject to } m \in M. \end{cases}$$

As we have already said, $K_1, \ldots, K_{N_Z}$ are continuous, Gateaux differentiable and convex, but not strictly convex. Then, we have to take $\lambda_C > 0$ (and considerer $f_j \in C([\underline{m}_j, \overline{m}_j])$ strictly convex for all $j = 1, \ldots, N_E$) in order to guarantee that $K_\lambda$ is continuous and strictly convex. In this situation, arguing in a similar way to [4], we can prove that problem (8) has only one solution. Then, if we define

$$(9) \quad \underline{S(P)} = \bigcup_{\substack{\lambda \in \mathbb{R}^{N_Z+1},\ \lambda \geq 0 \\ \lambda_C + \sum_{l=1}^{N_Z} \lambda_l = 1,\ \lambda_C > 0}} \{m \in M \ : \ m \text{ is the solution of } (8)\},$$

we have that each vector $m \in \underline{S(P)}$ is a Pareto-optimal solution of problem $(P)$. Moreover, if we assume that $f_j \in C^1([\underline{m}_j, \overline{m}_j])$ is convex for all $j = 1, \ldots, N_E$, functional $K_C$ is also continuous, Gateaux differentiable and convex. In this situation, if we denote by $S(P)$ the Pareto-optimal set of problem $(P)$ and

$$(10) \quad \overline{S(P)} = \bigcup_{\substack{\lambda \in \mathbb{R}^{N_Z+1} \\ \lambda \geq 0,\ \lambda_C + \sum_{l=1}^{N_Z} \lambda_l = 1}} \{m \in M \ : \ m \text{ is the solution of } (8)\},$$

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from Theorem 2 of [4], we know that $\underline{S(P)} \subset S(P) \subset \overline{S(P)}$.

In spite of the lack of strictly convexity for $\lambda_C = 0$, a complete characterization of $S(P)$ can be given. However, the difficulty for specifying it increases with the number of sensitive areas $N_Z$. We illustrate the method to obtain the characterization for the simplest cases $N_Z = 1$ and $N_Z = 2$.

### 3.1. *Characterization of $S(P)$ for the case $N_Z$=1*

In this situation, we can prove the following result:

**Theorem 3.1.** *We assume that $N_Z = 1$, and let $f_j \in C^1([\underline{m}_j, \overline{m}_j])$ be strictly convex for all $j = 1 \ldots, N_E$. If there exists a vector $m \in M$ verifying $K_1(m) = 0$, then the Pareto-optimal set of problem $(P)$ is*

$$S(P) = \underline{S(P)} \cup \{m_{ecol}\}\,. \tag{11}$$

**Proof.** The solution $m_{ecol}$ of (7) is clearly a Pareto-optimal solution of problem $(P)$. Moreover, for $\lambda_C > 0$ the problem (8) has only one solution and then, as it is well know, if $m \in M$ is the solution of problem (8), it is also a Pareto-optimal solution of problem $(P)$.
Reciprocally, as above commented, from Theorem 2 of [4], we know that, if $m \in S(P)$, then there exists $\lambda = (\lambda_C, \lambda_1) \geq 0$, $\lambda_C + \lambda_1 = 1$ such that $m \in M$ is the solution of problem (8) with $N_Z = 1$. If $\lambda_C > 0$ then $m \in \underline{S(P)}$; otherwise, if $\lambda_C = 0$ ($\Rightarrow \lambda_1 = 1$) then $m$ is a solution of problem

$$\begin{cases} \text{Minimize } K_1(m) \\ \text{suject to } m \in M, \end{cases}$$

and because of $m_{ecol} \in M$ verifies $K_1(m_{ecol}) = 0$, then $K_1(m) = 0$. Moreover, $m \in M$ is a Pareto-optimal solution of Problem $(P)$, and therefore $K_C(m) \leq K_C(m_{ecol})$. Problem (7) has only one solution, and then $m = m_{ecol}$, which concludes the proof. □

### 3.2. *Characterization of $S(P)$ for the case $N_Z$=2*

In this situation, for $i = 1, 2$, we define $M_i = \{m \in M \; : \; K_i(m) = 0\}$ and consider the sub-problems:

- Problem $(P_1)$ : Find $m \in M_2$ which minimizes $K_C(m)$ and $K_1(m)$.
- Problem $(P_2)$ : Find $m \in M_1$ which minimizes $K_C(m)$ and $K_2(m)$.

For problem $(P_1)$ - completely analogous for $(P_2)$ - we consider the weighting problems

$$\begin{cases} \text{Minimize } \lambda_C K_C(m) + \lambda_1 K_1(m) \\ \text{suject to } m \in M, \;\; K_2(m) = 0. \end{cases} \tag{12}$$

For $\lambda_C > 0$, problem (12) has the same properties that problem (7). Then we define

$$\underline{S(P_1)} = \bigcup_{\lambda_C > 0,\ \lambda_1 \geq 0,\ \lambda_C + \lambda_1 = 1} \{m \in M \ : \ m \text{ is the solution of } (12)\},$$

and, from Theorem 3.1, $S(P_1) = \underline{S(P_1)} \cup \{m_{ecol}\}$. Thus, we can write a complete characterization of $S(P)$:

**Theorem 3.2.** *We assume that $N_Z = 2$, and let $f_j \in C^1([\underline{m}_j, \overline{m}_j])$ be strictly convex for all $j = 1 \ldots, N_E$. If there exists a vector $m \in M$ verifying $K_1(m) = K_2(m) = 0$, then the Pareto-optimal set of problem $(P)$ is*

$$S(P) = \underline{S(P)} \cup S(P_1) \cup S(P_2) = \underline{S(P)} \cup \underline{S(P_1)} \cup \underline{S(P_2)} \cup \{m_{ecol}\} \tag{13}$$

**Proof.** It is obvious that $S(P) \supset \underline{S(P)} \cup S(P_1) \cup S(P_2)$. Moreover, if $m \in S(P)$ then there exists $\lambda = (\lambda_C, \lambda_1, \lambda_2) \geq 0$, $\lambda_C + \lambda_1 + \lambda_2 = 1$, such that $m \in M$ is the solution of problem (8) with $N_Z = 2$.

- If $\lambda_C > 0$, then $m \in \underline{S(P)}$.
- If $\lambda_C = 0$, $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$, arguing in a similar way to Theorem 3.1, we obtain that $m = m_{ecol}$.
- If $\lambda_C = \lambda_1 = 0$ ($\Rightarrow \lambda_2 = 1$), we deduce that $K_2(m) = 0$ and $m \in M_2$. Then, because of $m \in S(P)$, it verifies that $m \in S(P_1)$.
- If $\lambda_C = \lambda_2 = 0$, we deduce that $m \in S(P_2)$. □

## 4. Numerical Resolution

As a first step we need to describe a useful method to compute objective functionals $K_C$, $K_1, \ldots, K_{N_Z}$, and their gradients.

### 4.1. *Computation of objective functionals and their gradients*

From the affinity and differentiability of function $F$, for any $m = (m_1, \ldots, m_{N_E})$, we have

$$F(m) = \sum_{k=1}^{N_E} \langle D_k F, m_k \rangle + F(0), \tag{14}$$

with $\langle D_k F, m_k \rangle = s_{k|\cup_{i=1}^{N_Z} A_i \times (0,T)}$, where $s_k$ denotes the sensibility with respect to $m_k$, which is given as the solution of the following system:

$$
\left.
\begin{array}{ll}
\dfrac{\partial s_k}{\partial t} + \vec{u} \cdot \nabla s_k - \beta \Delta s_k \rho + \kappa s_k = \dfrac{1}{h} m_k \delta(x - P_k) & \text{in } \Omega \times (0,T), \\
s_k(0) = 0 & \text{in } \Omega, \\
\dfrac{\partial s_k}{\partial n} = 0 & \text{on } \Gamma \times (0,T).
\end{array}
\right\} \tag{15}
$$

Since system (15) is linear with respect to $m_k$, if we approach $L^\infty(0,T)$ by a $N$-dimensional space, the knowledge of the sensibilities corresponding to the elements of its basis gives us directly the gradient of $F$ and, consequently, a direct manner in order to compute $F$, without solving each time the state system (1). Implicitly, this fact assures us that we can evaluate the objective functions (and their gradients) always in a straight manner, avoiding us to resolve each time a partial differential equation system. We will show this in detail in below paragraphs.

We take $N \in \mathbb{N}$, $\Delta t = \frac{T}{N}$, and define discrete times $t^n = n\Delta t$, for $n = 0, \ldots, N$. We consider the approximation of $L^\infty(0,T)$ given by

$$L^{\Delta t} = \{ f^{\Delta t} \in L^\infty(0,T) \ : \ f^{\Delta t}{}_{|(t^{n-1},t^n]} \in P_0, \ n = 1, \ldots, N \}.$$

Let $C^{\Delta t} = \{e_1^{\Delta t}, \ldots, e_N^{\Delta t}\}$ be the basis of this space induced by the canonical basis of $\mathbb{R}^N$. For each $k = 1, \ldots, N_E$, $n = 1, \ldots, N$, the sensibility $s_k^n$ corresponding to element $e_n^{\Delta t} \in C^{\Delta t}$ is the solution of system:

$$
\left.
\begin{array}{ll}
\dfrac{\partial s_k^n}{\partial t} + \vec{u} \cdot \nabla s_k^n - \beta \Delta s_k^n + \kappa s_k^n = \dfrac{1}{h} e_n^{\Delta t} \delta(x - P_k) & \text{in } \Omega \times (0,T), \\
s_k^n(0) = 0 & \text{in } \Omega, \\
\dfrac{\partial s_k^n}{\partial n} = 0 & \text{on } \Gamma \times (0,T).
\end{array}
\right\} \tag{16}
$$

For any $m_k^{\Delta t} \in L^{\Delta t}$, denoting $m_k^n = m_k^{\Delta t}(t^n)$, we can write $m_k^{\Delta t} = \sum_{n=1}^N m_k^n e_n^{\Delta t}$. Taking now $m^{\Delta t} = (m_1^{\Delta t}, \ldots, m_{N_E}^{\Delta t}) \in \left(L^{\Delta t}\right)^{N_E}$ we have:

$$K_l(m^{\Delta t}) = \frac{1}{2\epsilon_l} \int_0^T \int_{A_l} \left( F(m^{\Delta t}) - \sigma_l \right)_+^2 dx \, dt.$$

From expression (14) we obtain that $F(m^{\Delta t}) = \sum_{k=1}^{N_E} s_k^{\Delta t} + F(0)$, where $s_k^{\Delta t}$ represents the sensibility corresponding to $m_k^{\Delta t}$ (solution of system (15) replacing $m_k$ by $m_k^{\Delta t}$). However, bearing in mind that (15) is a linear system, and taking into account that $m_k^{\Delta t} = \sum_{n=1}^N m_k^n e_n^{\Delta t}$, we have that $s_k^{\Delta t} = \sum_{n=1}^N m_k^n s_k^n$, and, consequently,

$$K_l(m^{\Delta t}) = \frac{1}{2\epsilon_l}\int_0^T \int_{A_l} \left(\sum_{k=1}^{N_E}\sum_{n=1}^{N} m_k^n s_k^n + F(0) - \sigma_l\right)_+^2 dx\,dt, \tag{17}$$

and, of course,

$$K_C(m^{\Delta t}) = \sum_{k=1}^{N_E}\sum_{n=1}^{N} \Delta t\, f_k(m_k^n). \tag{18}$$

Moreover, for each $i = 1, \ldots, N$, we have

$$\begin{aligned} \langle \frac{K_l}{\partial m_j}(m^{\Delta t}), e_i^{\Delta t}\rangle \\ &= \frac{1}{\epsilon_l}\int_0^T \int_{A_l} \left(\sum_{k=1}^{N_E}\sum_{n=1}^{N} m_k^n s_k^n + F(0) - \sigma_l\right)_+ s_j^i\,dx\,dt, \end{aligned} \tag{19}$$

$$\langle \frac{K_C}{\partial m_j}(m^{\Delta t}), e_i^{\Delta t}\rangle = \Delta t\, f_j'(m_j^i). \tag{20}$$

So, $F(0)$ is computed by solving the state system (1) with the equation's second member equal to zero, and sensibilities $s^n_{k|\cup_{i=1}^{N_Z} A_i\times(0,T)}$ are computed by solving the corresponding systems (16). The resolution of all these systems is made by an accurate method, combining characteristics for the time discretization with Lagrange $P_1$ finite elements for the space discretization. This method, which presents very satisfactory convergence properties, is completely detailed in [7]. Finally, once $F(0)$ and $s^n_{k|\cup_{i=1}^{N_Z} A_i\times(0,T)}$ have been computed, the value of $K_l$ (and of its gradient) can be straightly obtained by using standard quadrature rules for the integrals in expressions (17) and (19).

### 4.2. *Computation of Pareto-optimal solutions*

Problem (7), and also problem (12) for $\lambda_C > 0$, can be solved by any numerical method for constrained convex differentiable optimization problems (in this work we use an interior point algorithm introduced by Herskovits [8] and analyzed by Panier *et al.* [9]). However, as we will show in next section, these solutions (ecological-optimal solutions) can be also approximated by solving the problem (8) for *suitable* weights. Then, we focus our attention on the computation of sub-optimal set $\underline{S(P)}$. From a computational viewpoint, the identification of $\underline{S(P)}$ can be divided in two stages:

DOI: 10.1685/2010CAIM536

- Stage 1: We must fix the number $N_{max}$ of points of $\underline{S(P)}$ we are interested in, and we have to choose their corresponding *weights* $\{\lambda^1, \ldots, \lambda^{N_{max}}\} \subset \mathbb{R}^{N_Z+1}$ verifying $\lambda^i \geq 0$, $\lambda_C^i + \sum_{l=1}^{N_Z} \lambda_l^i = 1$, $\lambda_C^i > 0$, for $i = 1, \ldots, N_{max}$. It is well known (see, for instance, Section 3.1.2 of [5] and the references therein) that, depending on the characteristics of the problem, this stage can be very laborious. In this paper we have used an algorithm based on generating the family of weight vectors by splitting the interval $[0,1]$ in an homogeneous way, as described by Caballero *et al.* in [10].

- Stage 2: For each $i = 1, \ldots, N_{max}$, we have to solve the problem (8) taking $\lambda = \lambda^i$. In order to do it, we have implemented a spectral projected gradient algorithm (SPG), introduced by Birgin *et al.* in [11], which provides the global minimum of the problem. At each step, $K_{\lambda^i}$ and its gradient are computed from the expressions obtained in previous subsection.

To solve the complete problem, the computational effort required by the overall procedure depends on several variables (particularly, time/space discretization parameters, number $N_{max}$ of points of $\underline{S(P)}$ we are interested in, and, of course, the number of objective functions $N_Z+1$). For example, in next section we are going to present two experiences in a realistic situation posed in the estuary of Vigo (Spain). By using an average PC, we have needed about 5 minutes to obtain the Pareto optimal frontier for Experience I ($N_Z = 1$), and about 20 minutes for Experience II ($N_Z = 2$).

## 5. Numerical results

Problem $(P)$ has been solved in a realistic situation posed in the estuary of Vigo (see Fig. 2), one of the most populous and industrialized cities in the NW of Spain. We are going to present two different experiences:

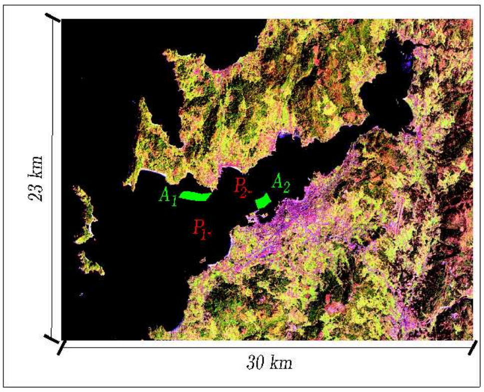


Fig. 2. Domain for numerical experiences: estuary of Vigo (NW Spain)

### 5.1. *Experience I: A unique sensitive area*

In this first example we have considered a unique purifying plant (located at $P_1$ in Fig. 2) and a unique sensitive area ($A_1$ in Fig. 2). We have considered a tidal cycle of 6 hours as period of time for simulation, null initial FC concentration ($\rho_0 = 0$), typical values for environmental impacts ($\sigma_1 = 0.0003$, $\epsilon_1 = 10^{-6}$) and depuration ($\underline{m_1} = 1$, $\overline{m_1} = 150$), and a standard cost function, given in Fig. 3: the depuration cost depends on the mass flow rate of FC discharged, in such a way that in order to discharge a lower amount of FC it is necessary to carry out a more intensive depuration, leading to a higher cost. Moreover, we have assumed $\overline{m_1} = 150$ for the FC concentration of the sewage arriving to the plant, thus the depuration cost above this value is constant and corresponds to the cost without depuration (100).

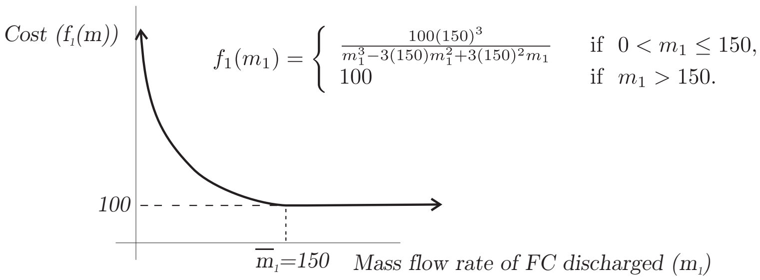


Fig. 3. Cost function for experience I

Table 1. Numerical results: $(a) = 10^{-6}\int_0^T f_1(m_1^{\Delta t})dt$, $(b) = 10^{-6}\int_0^T f_2(m_2^{\Delta t})dt$.

| $\lambda^i$ | $(a)$ | $(b)$ | $K_C(m^{\Delta t})$ | $K_1(m^{\Delta t})$ |
|---|---|---|---|---|
| (1,0) | 2.1600 | 2.1600 | $4.3200\times10^6$ | $13050.0\times10^6$ |
| (0.9,0.1) | 4.8215 | 4.1442 | $8.9658\times10^6$ | $5.5987\times10^6$ |
| (0.7,0.3) | 5.1874 | 4.5290 | $9.7164\times10^6$ | $1.1997\times10^6$ |
| (0.5,0.5) | 5.3654 | 4.7330 | $10.098\times10^6$ | $0.4552\times10^6$ |
| (0.3,0.7) | 5.5051 | 4.9091 | $10.414\times10^6$ | $0.1761\times10^6$ |
| (0.1,0.9) | 5.6823 | 5.1334 | $10.816\times10^6$ | $0.0384\times10^6$ |
| (0.075, 0.925) | 5.7095 | 5.1712 | $10.881\times10^6$ | $0.0273\times10^6$ |
| (0.050, 0.950) | 5.7475 | 5.2232 | $10.971\times10^6$ | $0.0174\times10^6$ |
| (0.025, 0.975) | 5.7981 | 5.2978 | $11.096\times10^6$ | $0.0086\times10^6$ |
| (0.005, 0.995) | 5.8887 | 5.4152 | $11.304\times10^6$ | $0.0018\times10^6$ |
| (0.00001,0.99999) | 6.0335 | 5.5626 | $11.596\times10^6$ | $0.00001\times10^6$ |

The Pareto-optimal frontier (with a close-up on lowest values of $K_1$) is shown in Fig. 4. In Table 1 we show the values of objective functions for

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some Pareto-optimal solutions (elements of $\underline{S(P)}$ obtained as described in previous section). As we can see, if $\lambda^i$ is close to $(0, 1)$, then the value of $K_1$ is small, and Pareto-optimal solutions close to the ecological-optimal solution $m_{ecol}$ are expected. To make sure of this, we compute $m_{ecol}$ by solving the problem (7) in this particular case, and compare it with the Pareto-optimal solution corresponding with $\lambda^i = (0.00001, 0.99999)$. The similarities can be seen in Fig. 5.

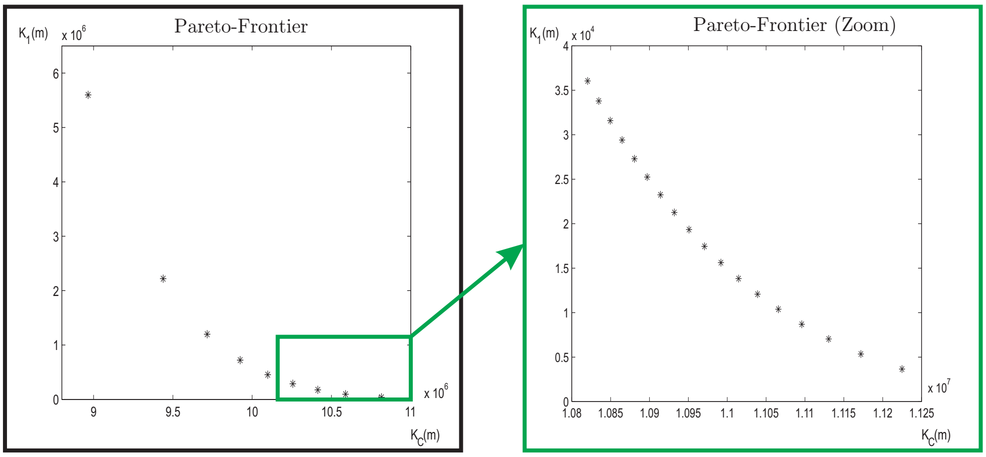
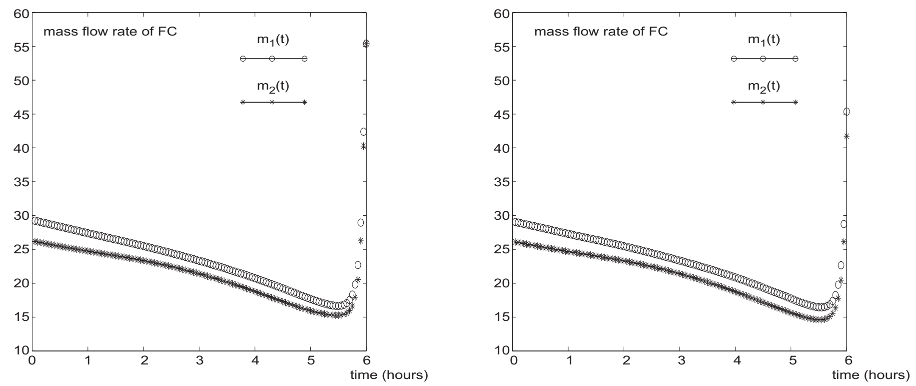



Fig. 4. Pareto-optimal frontier for experience I




Pareto-optimal solution corresponding with $\lambda$=(0.00001,0.99999)    Ecological optimal solution $m_{ecol}$

Fig. 5. Comparison of optimal strategies

### 5.2. *Experience II: Two sensitive areas*

In this second experience we have considered another purifying plant, located at $P_2 \in \Omega$, and also another sensitive area $A_2$ (see Fig. 2). We have supposed that $A_2$ is less sensitive than $A_1$, and then we have taken $\sigma_2 = 0.0005$. For the rest of parameters we have considered the same values that in previous experience (particularly, $\epsilon_2 = \epsilon_1 = 10^{-6}$, $\underline{m_2} = \underline{m_1} = 1$, $\overline{m_2} = \overline{m_1} = 150$, and $f_2 = f_1$ given in Fig. 3. In this case, we have three objectives and the Pareto-optimal frontier becomes a surface. It has been obtained from determining $\underline{S(P)}$ as described in previous section, and the results can be seen in Fig. 6. Finally, we point out, as a practical application, two useful utilities of this surface:

1. If the plant manager gives a fixed value for purifying cost (say $K_C^{max}$), the surface can be intersected with the horizontal plane $K_C = K_C^{max}$, and it provides a curve determining, for the same cost, several Pareto-optimal solutions that will favour (from an environmental viewpoint) one area over another. Among the different optimal solutions, the plant manager should choose one of them invoking, for instance, commercial or political criteria.
2. If the plant manager gives fixed values for environmental impacts in each plant ($K_1^{max}$, $K_2^{max}$), the surface can be intersected with vertical planes $K_1 = K_1^{max}$ and $K_2 = K_2^{max}$ and a point determining the *cheapest* Pareto-optimal solution is obtained.

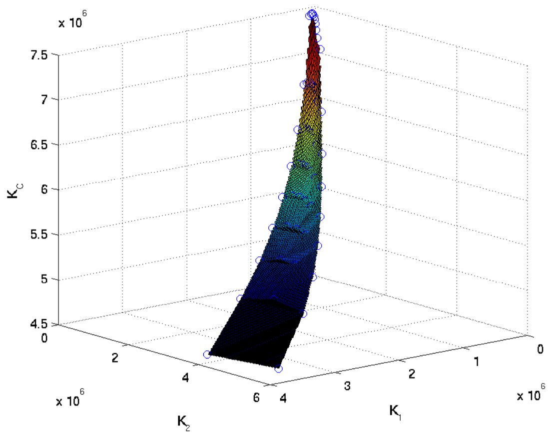



Fig. 6. Pareto-optimal frontier for experience II

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