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Thermodynamical equilibrium and space-time geometry - a survey

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Abstract

Abstract Based on the balance equations for energy-momentum, spin, particle and entropy density, an approach is considered which represents a framework for special- and general-relativistic continuum thermodynamics. A general entropy 4-vector, containing particle, energy momentum, and spin density contributions, is introduced. In a second part, we confine this framework to General Relativity Theory and show exemplarily that the incorporation of gravitational field equations has strong consequences for relativistic continuum thermodynamics.

Keywords: `Cosmology, relativistic theory of irreversible processes, thermodynamic equilibrium, conformal Killing vector`

AMS Subject Classification: `80A10, 80A17, 83C55`

1. Equilibrium in an extended version of the Eckart Theory of Irreversible Processes

1.1. *The framework*

The Eckart approach to the so-called relativistic Theory of Irreversible Processes [1] (see also [2–5]) is based on relations that are generated by a transfer of the non-relativistic field formulation of Irreversible Continuum Thermodynamics into a Lorentz-covariant form from which its general-covariant version can be obtained as usual[a]. The primary ingredients of the covariant theory are the basic dynamical variables: the particle flux 4-vector N^k, the (symmetric) energy-momentum tensor T^{ik} and the entropy

[a]For proofs and other details of this Chapter, see [6] and the literature cited therein.

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vector S^k, satisfying the balance relations

$$N^k{}_{;k} \;=\; 0, \qquad T^{ki}{}_{;k} \;= 0 \tag{1}$$

and the dissipation inequality

$$S^k{}_{;k} \;\geq\; 0. \tag{2}$$

These balance equations can be interpreted in a two-fold way: Firstly, remaining in the framework of Special Relativity Theory, i.e., not including gravitation, these relations furthermore live in the Minkowski space. Then the covariant derivative denoted by the semicolon only says that the relations are not referred to Cartesian coordinates and inertial systems, respectively. Secondly, one can interpret these relations as balances in a curved pseudo-Riemannian space-time, whose curvature is determined by gravitation. Then there are no global Cartesian coordinate and inertial systems, but only non-Cartesian (curvilinear) coordinates and non-inertial reference systems such that the covariant derivative is a matter of necessity. To complete the theory one had to incorporate Einstein's gravitational equations and matter (constitutive) equations.

In this part, however, we are not dealing with definite complete gravitational theories and specified matter (constitutive) equations. Rather we are here concerned with an extension of Eckhart's framework to the case of balance equations which firstly are valid in Minkowski, Riemann and post-Riemann geometries, which secondly regard spin as well as production and supply terms in the energy-momentum and spin balances and which thirdly are not necessarily restricted to near-equilibrium situations.

Thus, for a 1-component material, we start out with the balance equations of the particle flux 4-vector N^k, the energy-momentum tensor T^{ik} and the spin tensor $S_{ji}{}^k$:

$$N^k{}_{;k} \;=\; 0, \qquad T^{ki}{}_{;k} \;=\; G^i + K^i, \qquad S_{ji}{}^k{}_{;k} \;=\; H_{[ji]} + L_{[ji]} \tag{3}$$

[b]and with the balance of the 4-entropy S^k:

$$S^k{}_{;k} \;=\; \varphi + \sigma, \qquad \sigma \geq 0. \tag{4}$$

As usual, the semicolon "**;**" denotes the covariant derivative, T^{ik} is the energy momentum-tensor of a material which is not necessarily symmetric with vanishing covariant derivative, the spin tensor $S_{ji}{}^k$ is skew-symmetric

[b]Square brackets are also used to emphasize that the tensor is antisymmetric, $L_{(ji)} = 0$, especially for H_{ji} and L_{ji}.

in the lower indices, and φ is the entropy supply. The force K^i and the angular momentum $L_{[ji]}$ are the external sources of the energy momentum tensor and of the spin tensor.

The supply terms φ, K^i, and $L_{[ji]}$ are substitutes for cases in which the energy-momentum tensor and/or the spin tensor do not include all fields, so that additional fields come into account by external sources. Of course, in a field theory describing systems completely by the equations of the fundamental fields, external sources do not occur. If one is forced to introduce supply terms, this shows that the theory is not field-theoretically complete. To complete it, one has to describe the supply terms by additional fundamental fields in such a way that they can be absorbed by the other expressions in the balances (3). The G^i and $H_{[ij]}$ are internal source terms caused by the choice of a special space-time and by the spin-momentum-energy coupling (SMEC). For instance, in Einstein-Cartan geometry, the G^i and $H_{[ij]}$ are caused by the torsion and depend as coupling terms on the energy-momentum and on the spin tensor. We call a theory for which the G^i and $H_{[ij]}$ vanish identically SMEC-free.

1.2. *The entropy vector*

In literature, one finds different approaches to a special- and general-relativistic conception of entropy. Most of them is in common that entropy is described by a 4-vector, but there are proposed different expressions for it (which generally do not incorporate spin terms). To decide the question as to the correct entropy vector we make use of the following identity: Independently of the special interpretation of the time-like vector field u^k, the following identity for the 4-entropy is valid:

$$(5) \quad S^k \;\equiv\; (s^k - \lambda q^k - \mu n^k - \Lambda^m \Xi_m^{\cdot k}) + (\mu N^k + \xi_l T^{kl} + \zeta^{nm} S_{nm}{}^k),$$

with the following abbreviations:

$$(6) \qquad \lambda \text{ arbitrary scalar}, \; \Lambda^k \text{ arbitrary tensor field of 1st order},$$

$$(7) \quad \mu \;:=\; \frac{1}{n}(s - \lambda e - \Lambda^m \Xi_m), \quad \xi_l \;:=\; \lambda u_l, \quad \zeta^{nm} := 2u^n \Lambda^p h_p^m$$

(s^k and n^k are projections of S^k and N^k to the space orthogonal to the 4-velocity u^k, $h_l^k = \delta_l^k - \frac{1}{c^2} u^k u_l$ is the projector, $\Xi_k{}^l := 2S_{ab}{}^c h_c^l u^a h_k^b$ is the spin stress, $q^k := h_l^k u_m T^{lm}$ the heat flux, s the entropy density and e the energy density). Two different interpretations of the u^k can be found in literature: the first one is due to Landau-Lifshitz [7], the second one due to Eckart [1].

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Here we prefer Eckart's choice:

$$u^k := \frac{a^2}{n}N^k, \quad a \equiv c, \quad \text{or} \quad n^k \equiv 0, \tag{8}$$

for this choice does not restrict the energy-momentum tensor and the spin tensor, either. We now introduce Eckart's version into the entropy identity (5)

$$S^k \equiv (s^k - \lambda q^k - \Lambda^m {\Xi_m}^k) + (\mu N^k + \xi_l T^{kl} + \zeta^{nm} {S_{nm}}^k), \tag{9}$$

where (6) and (7) are still valid.

In order to determine the entropy vector in accordance with this identity, one can exploit the entropy balance (4)

$$S^k{}_{;k} = \varphi + \sigma. \tag{10}$$

In particular, this leads to

$$\lambda = \frac{1}{T}, \qquad \Lambda^k = \frac{1}{c^2}\frac{s_{lm}u_i}{T}\Theta^{[lm][ik]}. \tag{11}$$

The tensor s_{lm} denotes the spin density, while $\Theta^{[lm][ik]}$ connects the spin to the momenta and does not need to be specified here for our purposes. Defining now the entropy flux by

$$s^k := \frac{1}{T}q^k + \Lambda^m {\Xi_m}^k \tag{12}$$

the entropy production results in

(13)

$$\sigma = \frac{1}{T}u_l G^l + \frac{2}{T}u^{[i}\Lambda^{k]}H_{[ik]} + \mu_{,k}N^k + (\frac{1}{T}u_l)_{;k}T^{kl} + 2(u^{[n}\Lambda^{m]})_{;k}{S_{nm}}^k \geq 0$$

and the entropy vector reads

$$S^k = \mu N^k + \frac{1}{T}u_l T^{kl} + 2u^{[n}\Lambda^{m]}{S_{nm}}^k. \tag{14}$$

1.3. *Equilibrium conditions*

The necessary equilibrium conditions (marked in the following by $\doteq$) are given by vanishing entropy production density and vanishing entropy supply density

$$\sigma_{eq} \doteq 0, \quad \varphi_{eq} \doteq 0 \quad \longrightarrow \quad S^k{}_{;k}{}^{eq} = 0. \tag{15}$$

(equilibrium quantities are marked by $_{eq}$ or by ^{eq} in the sequel) and vanishing entropy flux density

$$s^k{}_{eq} \doteq 0. \tag{16}$$

Exploiting this we obtain

$$u_i{}^{eq} K^i{}_{eq} = 0, \qquad 2(u^{[i}\Lambda^{k]})_{eq} L_{[ik]}{}^{eq} = 0. \tag{17}$$

From (17) we read off, that for the present neither the external forces nor the external momenta have to be zero in equilibrium. Using the balance equations $(3)_{2,3}$, we obtain

$$u_i{}^{eq}[T^{ki}{}_{;k} - G^i]_{eq} = 0, \qquad (u^{[i}\Lambda^{k]})_{eq}[S_{ik}{}^j{}_{;j} - H_{[ik]}]_{eq} = 0. \tag{18}$$

From (14) follows by $(15)_3$

$$0 = \Big(\mu N^k\Big)^{eq}_{;k} + \left(\frac{1}{T} u_l T^{kl}\right)^{eq}_{;k} + 2\left(u^{[n}\Lambda^{m]} S_{nm}{}^k\right)^{eq}_{;k}. \tag{19}$$

The N^k, T^{kl} and $S_{nm}{}^k$ are not independent of each other, because they are coupled by constitutive equations and by the SMEC-terms. Therefore, we cannot state that each term of the sum (19) vanishes. The equilibrium condition (19) is only one equation which cannot describe equilibrium completely. Therefore, we need supplementary equilibrium conditions beyond (15) and (16). This is not surprising since our quasi-axiomatic approach is not complete. In [6] supplementary conditions are introduced such that the equilibrium is determined uniquely.

2. Equilibrium in a complete framework: the Theory of Irreversible Processes and General Relativity Theory

2.1. *The framework*

Now[c] we complete the above-given framework by supplementing the balances by Einstein's gravitational field equations that assume a pseudo-Riemannian space-time with a corresponding Levi-Civita derivative. As a consequence, the energy-momentum tensor is symmetric, there is no spin balance, and in the remaining energy-momentum balance there are neither

[c]For proofs and other details of this Chapter, see Ref. [8] and the literature cited therein. In contrast to the previous chapter, here and in the cited literature we use the signature +2 and put $c = 1$.

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internal nor external sources. That means, we start from Einstein's equations

$$R_{ab} - \frac{1}{2} g_{ab} R = -\kappa T_{ab} \tag{20}$$

and the relations (1) and (2), where the entropy vector is given by

$$S^k \; = \; \mu N^k + \frac{u_m}{T} T^{km}. \tag{21}$$

The vanishing of the covariant divergence of this vector is again assumed to be a necessary equilibrium condition.

Taking into account Riemannian geometry and Einstein's field equations, the condition $S^k{}_{;k} = 0$ implies: u^a/T is a Killing or conformal Killing field.

$$\left(\frac{u_a}{T}\right)_{;b} + \left(\frac{u_b}{T}\right)_{;a} = \alpha(x^c) g_{ab}. \tag{22}$$

2.2. *Fluid dynamics*

It is possible to decompose the energy-momentum tensor T_{ab} with respect to the velocity field

$$T_{ab} = e u_a u_b + p h_{ab} + 2u_{(a} q_{b)} + \pi_{ab}. \tag{23}$$

Here $e = T_{ab} u^a u^b$ denotes the energy-density, $p = \frac{1}{3} T_{ab} h^{ab}$ the pressure, $q_a = -T_{cb} u^b h^c{}_a$ the heat-flow and $\pi_{ab} = \pi_{ba} = T_{cd} h^c{}_a h^d{}_b - p h_{ab}$ the anisotropic pressure. From these definitions follow the orthogonality conditions $q_a u^a = 0$ and $\pi_{ab} u^a = 0$ and, moreover, $\pi^a{}_a = 0$.

To derive the dynamics of the fluid one has to exploit (for notation see the end of the paragraph):
The energy balance

$$0 = \dot{e} + (e+p)\Theta + q^a{}_{;a} + \dot{u}_a q^a + \sigma_{ab} \pi^{ab} \tag{24}$$

and the momentum balance

$$0 = \dot{u}_a (e+p) + h_a{}^b (p_{;b} + \dot{q}_b + \pi_b{}^c{}_{;c}) + (\omega_{ab} + \sigma_{ab} + \frac{4}{3}\Theta h_{ab}) q^b \tag{25}$$

both stemming from (1) with (23). Furthermore, it is useful to remember the decomposition of the Riemann tensor in its Ricci and Weyl C_{abcd} parts

$$R^{ab}{}_{cd} = C^{ab}{}_{cd} + 2g^{[a}{}_{[c} R^{b]}{}_{d]} - \frac{R}{3} g^a{}_{[c} g^b{}_{d]}, \tag{26}$$

and the contracted Bianchi identity

$$C^{abcd}{}_{;d} = R^{c[a;b]} - \frac{1}{6} g^{c[a} R^{;b]}. \tag{27}$$

Furthermore, using the Ricci identity for the velocity field u^a

$$u_{a;b;c} - u_{a;c;b} = R_{adcb} u^d. \tag{28}$$

Introducing the kinematic invariants by the decomposition and implying further information that give Einstein's equation beyond (1), finally one obtains propagation and constraint equations for those invariants describing the dynamics of the fluid.

$$u_{i;n} = \omega_{in} + \sigma_{in} + \frac{\Theta}{3} h_{in} - \dot{u}_i u_n. \tag{29}$$

Here the space-like antisymmetric part $\omega_{in} = h_i^a h_n^b u_{[a;b]} = u_{[i;n]} + \dot{u}_{[i} u_{n]}$ denotes the *rotation* of the flow of the fluid, the space-like symmetric traceless part $\sigma_{in} = h_i^a h_n^b u_{(a;b)} - \frac{\Theta}{3} h_{in} = u_{(i;n)} + \dot{u}_{(i} u_{n)} - \frac{\Theta}{3} h_{in}$ the *shear* (the volume of the elements are conserved, but the shape is distorted), $\Theta = u^i_{;i}$ is the *expansion* (change of volume), $\dot{u}_i = u_{i;n} u^n$ the *acceleration* of the velocity field, and $h_{in} = g_{in} + u_i u_n$ is the projector on the space orthogonal to u^n.

With this knowledge one can utilize the (conformal) Killing equation (22) satisfied by the temperature vector. The analysis provides the following results (all the following equations are contain equilibrium quantities):

Killing case: From the Killing equation it follows for the temperature

$$\left(\frac{1}{T}\right)_{,k} u^k \;=\; 0 \tag{30}$$

(temperature is time-independent), for the expansion and shear

$$\Theta \;=\; 0, \qquad \sigma_{lk} \;=\; 0 \tag{31}$$

(expansion-free, shear-free) and for acceleration

$$\dot{u}_a \;=\; 0 \tag{32}$$

or

$$\dot{u}_a \;=\; T h_a{}^m \left(\frac{1}{T}\right)_{,m} . \tag{33}$$

Furthermore, the balance equations provide

$$\dot{e} \;=\; 0 \qquad \dot{p} \;=\; 0 \tag{34}$$

(energy-density and pressure time-independent) such that the two cases of $\dot{u}_a = 0$ and $\dot{u}_a \neq 0$ lead to the following conditions:
(i) vanishing acceleration

$$\omega_{ik} = 0, \tag{35}$$

(ii) non-vanishing acceleration

$$\begin{aligned} h_i{}^k \dot{q}_k &= \omega_{ik} q^k \\ h_i{}^k h_j{}^m \dot{\pi}_{km} &= 2\pi^k{}_{(i}\omega_{j)k} \end{aligned} \tag{36}$$

(Cattaneo-type equations).

Conformal Killing case: The conformal Killing equation leads to

$$\left(\frac{1}{T}\right)_{,k} u^k \neq 0, \quad \Theta \neq 0. \tag{37}$$

For a further analysis of this eqution one has again to distinguish the two cases:
(i) vanishing acceleration

$$\Theta = 0, \qquad \sigma_{lk} = 0 \tag{38}$$

(expansion-free, shear-free or Friedmann-Robertson-Walker type space-times),
(ii) non-vanishing acceleration

$$h^b_a \dot{q}_b = T\dot{\alpha}_{,b} h^b_a - T\alpha_{,m}(\omega^m{}_a + \frac{1}{3}\Theta h^m_a) - \frac{2\Theta}{3} q_a - q^k \omega_{ka} \tag{39}$$

$$\begin{aligned} h^m{}_a h^b{}_c \dot{\pi}_{bm} = &-\frac{T}{2} h_{ac}\Box\alpha - T h^m_a h^b_c \alpha_{;m;b} + h_{ac}\frac{\dot{p}-\dot{e}}{2} + \\ &+2\pi^k{}_{(a}\omega_{c)k} - \frac{2\Theta}{3}\pi_{ac} + \frac{\Theta(p-e)}{3} h_{ac} \end{aligned} \tag{40}$$

(Cattaneo-type equations).

3. Conclusion

If one intends to define thermodynamical equilibrium by exploiting the necessary condition of vanishing divergence of the entropy density 4-vector, this vector has to be determined correctly. Using an entropy identity together with Eckart's ansatz of the 4-velocity this could be done in the general case, where the spin of matter is incorporated. As a generalization of Eckart's special-relativistic theory of irreversible thermodynamical processes, we considered the question of equilibrium for this more general case of non-vanishing spin in a geometrically extended framework. This framework does not confine to the special-relativistic Minkowski space-time, but allows also for (curved) Riemann space-times and post-Riemann (i.e., Riemann-Cartan) space-times. As in the Minkowski case, the equilibrium conditions obtained from the general-covariantly written balance equations have to be supplemented by conditions that have to be motivated by appropriate physical considerations. It was shown how this can be done without introducing matter equations that restrict the class of materials. Of course, as far as the necessity of supplementary conditions is concerned, here one meets a similar situation as special relativity: In both cases, the balance equations do not determine the dynamics uniquely. In special relativity, this is inevitable for it is the general frame for a dynamics with a fixed space-time background, and the balances are only necessary conditions of a dynamics. In the more general case of curved space-times, the system is even yet less determined for we did not assume gravitational field equations specifying the space-time structure.

What happens when one includes gravitational equations determining the space-time structure exemplarily was seen in the case of general relativity. Then one has to assume a Riemann space-time with the covariant Levi-Civita derivative and Einstein's gravitational equations which imply corresponding balances. It was shown that in equilibrium (i) the temperature vector has to be a Killing or a conform Killing vector (in special relativity a set of supplementary equilibrium conditions had to be added by hand to prove that the temperature vector is a Killing vector), (ii) as an implication of the Killing property of the temperature vector, most assumptions made can be derived, without restricting the matter configuration to a perfect fluid, (iii) for non-vanishing rotation of the fluid, the heat-flow is unequal to zero, (iv) for vanishing acceleration of the fluid the Friedmann radiation cosmos is the only physically significant solution of Einstein's equations and (v) the equilibrium conditions are of the Cattaneo form such that a causal propagation of temperature can be expected.

REFERENCES

1. C. Eckart, Thermodynamics of irreversible processes iii, 58 (1940) 919., *Phys. Rev.*, vol. 28, pp. 919–924, 1940.
2. W. Israel, Covariant fluid mechanics and thermodynamics: An introduction, in *Relativistic Fluid Mechanics* (A. Anile and Y. Choquet-Bruhat,

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eds.), pp. 152–210, Springer-Verlag, 1989.

3. S. Weinberg, *Gravitation and Cosmology.* Wiley, 1972.
4. E. Calzetta, Relativistic fluctuating hydrodynamics, *Class. Quantum Grav.*, vol. 15, pp. 653–667, 1998.
5. G. Neugebauer, *Relativistische Thermodynamik.* Akademie-Verlag, 1980.
6. W. Muschik and H.-H. v. Borzeszkowski, Entropy identity and material-independent equilibrium conditions in relativistic thermodynamics, *Gen. Rel. Grav.*, vol. 41, pp. 1285–1304, 2009.
7. L. Landau and E. Lifshitz, *Fluid Mechanics.* Addison-Wesley, 1959.
8. T. Chrobok and H.-H. v. Borzeszkowski, Thermodynamical equilibrium and spacetime geometry, *Gen. Rel. Grav.*, vol. 38, pp. 397–415, 2006.