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Statistical mechanics and thermodynamics of turbulent quantum vortex tangles

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Abstract

In this paper we present some phenomenological ideas about the thermodynamics of quantized vortex loops arising in superfluid turbulence. The system of vortex loops may be seen as a dissipative structure, not existing on its own but only under the influence of an external heat flux. Starting from a simple definition of the temperature of the vortex tangle and from the relation between energy and vortex length, we obtain the entropy of the system, as well as the caloric and thermal equations of state, relating internal energy and pressure to temperature and volume. We discuss the connection between our macroscopic results and microscopic results on vortex length distribution function having a potential form.

Keywords: Superfluid turbulence; Statistical thermodynamics.

AMS Subject Classification: 82D50; 82B30

1. Introduction.

Turbulent superfluids sustain a very complex tangle of vortex loops. This tangle is produced and sustained by a heat flux crossing the system. When such a heat flux, or counterflow velocity, is high enough, vortices appear and evolve; when the heat flux is too small, vortices do not appear. Thus, the vortex tangle is a kind of dissipative structure, in the sense of Prigogine, namely, it is conceptually analogous to the convective cells appearing in the Bénard problem in a viscous fluid submitted to a temperature gradient [1], [2], but it is far more complex than them from the geometrical and the dynamical points of view. Here, we discuss some thermodynamic characteristics of the vortex tangle as if it was a system in internal equilibrium with itself but out of equilibrium with the underlying superfluid helium, due to the continuous energy exchange between them. We may try a macroscopic description of this system because its microscopic dynamics –concerning crossing, breaking and recombining vortex loops– is much

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faster than the time scale of typical macroscopic observations.

In superfluid turbulence the vortex line structure is a disordered tangle of lines. A fundamental scalar quantity used for its characterization is the vortex line length per unit volume, L, briefly called line density. Line density L may be directly measured, for instance, by means of second sound or temperature gradient and is therefore the most used variable describing the tangle [3]– [7]. Experimental data in stationary situations show that in this case line density L is proportional to the square of the heat flux $\mathbf{q}$, i.e. to the square of the counterflow velocity $\mathbf{V}_{ns} = \mathbf{v}_n - \mathbf{v}_s$ ($\mathbf{v}_n$ and $\mathbf{v}_s$ being the mesoscopic velocities of the normal and the superfluid component of helium II) [3]– [7]:

$$L = L_H = \gamma \frac{V_{ns}^2}{\kappa^2}. \tag{1}$$

Here γ is a dimensionless constant dependent on the temperature of the helium.

In the microscopic model of superfluid turbulence [3,4], [8], the vortex lines are described by a vectorial function $\mathbf{s}(\xi, t)$, where ξ is the arc-length. The primes indicate differentiation with respect to ξ, $\mathbf{s}' = \partial \mathbf{s}/\partial \xi$ is the unit vector tangent to the vortex and $\mathbf{s}'' = \partial^2 \mathbf{s}/\partial \xi^2$ the curvature vector.

In the present work we deal with the thermodynamic aspects arising from relevant features of the geometric structure of the tangle, in particular, the vortex length distribution of the vortex loops. Indeed, the vortices must be either open loops, or pinned loops, with their ends on the walls of the container. When turbulence is completely developed, i.e. for relatively high values of the heat flux or the counterflow velocity, the tangle consists mainly of free loops, continuously colliding amongst them, breaking and recombining, and and may also remain a few pinned loops. In contrast, for small values of the heat flux pinned vortex lines predominate, but we do not consider here this latter situation.

The plan of the paper is the following: in Section 2 we define the temperature of the vortex tangle and, from this definition, we express the internal energy in terms of this temperature, and we obtain the entropy and the pressure of the system; in Section 3, we show that the macroscopic results agree with those obtained on microscopic grounds from a vortex length distribution function of potential form proposed in the literature. In Section 4, we discuss the stability and comment on an analogy with black hole thermodynamics.

2. Temperature, pressure and entropy of the vortex tangle

We note that the tangle is not a continuous line but a superposition of many closed vortex loops and pinned vortices, which continuously collide, cross, break and recombine. We will consider fully developed counterflow superfluid turbulence, in which case closed vortex loops are much more abundant than pinned vortices (recall that the vortex lines must be either closed on themselves, or must have their ends attached to the walls of the container [3]– [7]) and we will suppose that the line density L is sufficiently high, in such a way that we can neglect the total length of pinned vortices. In this case, being the tangle composed of a very large number of closed vortex loops, one might consider the tangle as a gas of vortex loops.

The two main hypotheses of the model are to assume that the temperature T of the system of closed loops is given by

$$k_B T = < U_l >, \tag{2}$$

where U_l is the energy of a loop of length l, k_B is the Boltzmann's constant, and $< ... >$ denotes the average over the loops of different lengths, and to assume that the average length of the loops behaves as

$$< l >= \lambda L^{-1/2}, \tag{3}$$

L being the total length of loops per unit volume, which has dimensions of $(length)^{-2}$, and λ a dimensionless constant. This hypothesis follows from microscopic analysis [9] which will be commented at the end of the letter and is related to the statistical properties of the system of loops. In physical terms, equation (3) implies that the average length and curvature radius of loops are of the order of their average separation, which is given by $L^{-1/2}$. Note that equation (3) implies that the total number of loops per unit volume N_V, given by $N_V = L/ < l >$, will behave as $N_V = \lambda^{-1} L^{3/2}$. Thus, a reduction of the average size of the loops (higher L) will imply a big increase of the number of loops, and viceversa.

In the case of vortex loops, it is usually assumed that their energy is proportional to their length [3]– [7]. Here, for the sake of exploring a wider range of physical possibilities, we assume that the energy U_l of a line of length l is

$$U_l = a_0 l^{1+\alpha}, \tag{4}$$

with α a constant exponent, and a_0 a constant given by $a_0 = \epsilon_V l_0^\alpha$, ϵ_V being the vortex tension, whose dimensions are energy per unit length, and l_0 a fixed reference length.

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The term α in the exponent of the expression (4) describes possible deviations with respect to the linear behaviour in the energy-length relation. A deviation with respect to the linear behaviour may be attributed to the interaction between the several parts of the vortex lines; for instance, a source of discrepancy with respect to the linear expression may be the presence of kinks, arising from recombinations of previous loops; if the kink is very acute, the two parts of it may have a strong interaction between themselves, thus leading to contributions which would be absent in a smooth model of lines. Studies on superfluid and classical turbulence suggest that the coefficient α could not be an integer, that is the interactions between loops or between other parts of the same loop could contribute to fractal aspects in such a way that the contribution to the energy is not linear nor quadratic [10].

Here, we will not discuss the physical origin of the parameter α, but its influence on the macroscopic equations of state. It could be guessed, however, that a positive (negative) value of α corresponds to situations where long wavelength perturbations have a higher (lower) contribution to the energy than short wavelength perturbations. Indeed, an exponent higher than 1 in $U_l \sim l^{1+\alpha}$ gives proportionally a higher increase of energy when l increases, and the opposite follows for exponents lower than 1 [10].

Thus, relation (2) and (4) lead to $a_0 < l^{1+\alpha} >= k_B T$. In view of (3), and of $< l^{1+\alpha} >= \beta < l >^{1+\alpha}$, with β being a numerical constant which depends on the detailed form of the statistical distribution of the lengths of the loops, which will be discussed in Section 3, temperature scales with L as

$$k_B T = a_0 \beta \lambda^{1+\alpha} L^{-(1+\alpha)/2} \equiv A_T L^{-(1+\alpha)/2}, \tag{5}$$

with $A_T = a_0 \beta \lambda^{1+\alpha}$. This implies that the energy density per unit volume U_V depends on T as

$$U_V = k_B T N_V = \frac{A_T}{\lambda} L^{(2-\alpha)/2} = \frac{A_T}{\lambda} \left(\frac{k_B}{A_T} \right)^{-\frac{2-\alpha}{1+\alpha}} T^{-\frac{2-\alpha}{1+\alpha}}. \tag{6}$$

Since, according to the fundamental Gibbs equation of thermodynamics, the entropy per unit volume S_V satisfies the relation $(\partial S_V / \partial U_V) = T^{-1}$, we obtain

$$\frac{\partial S_V}{\partial L} = \frac{\partial S_V}{\partial U_V} \frac{\partial U_V}{\partial L} = \frac{1}{T} \frac{A_T}{\lambda} \frac{2-\alpha}{2} L^{-\alpha/2} = \frac{k_B}{\lambda} \frac{2-\alpha}{2} L^{1/2}. \tag{7}$$

Integrating this equation we get for the entropy per unit volume $S_V(L)$

$$S_V(L) - S_0 = k_B \frac{2-\alpha}{3\lambda} (L^{3/2} - L_0^{3/2}). \tag{8}$$

It is worth noting that the exponent of L in this expression does not depend on α. This implies that the value of α will not have an influence in an adiabatic expansion (i.e. at constant total entropy). Furthermore, to have a positive entropy requires α to be less or equal than 2.

Since $S_V(L)$ is the contribution of the loops to the entropy, it is logical to expect that when $L = 0$, this contribution to the entropy will be 0; therefore, one can ignore the additive factor in (8), i.e. one may set $S_0 = 0$ and $L_0 = 0$.

To find the expression of the pressure p we need the total entropy $S(U,V)$, to apply the well known thermodynamic relation $(\partial S/\partial V)_U = p/T$ [11]. We assume that $S(U,V) \equiv S_V(U/V)V$, in other words, that entropy (as well as energy) are extensive quantities, namely that, for an uniform system, the total energy and total entropy are equal to their respective densities per unit volume times the total volume, i.e. $U = U_V V$, $S = S_V V$.

In view of the relation (6) between U_V and L, and of the extensivity hypothesis that we have just discussed, neglecting, as it has been said, the constant of integration in (8), we may write the total entropy as

(9) $$S(U,V) = k_B \frac{2-\alpha}{3\lambda} L^{3/2} V = k_B \frac{2-\alpha}{3\lambda} \left[\frac{\lambda}{A_T}\right]^{\frac{3}{2-\alpha}} \left(\frac{U}{V}\right)^{\frac{3}{2-\alpha}} V.$$

From expression (9), the pressure may be obtained from the well known thermodynamical relation $(\partial S/\partial V)_U = p/T$, leading to

(10) $$\frac{p}{T} = -k_B \frac{\alpha+1}{3\lambda} \left[\frac{\lambda}{A_T}\right]^{\frac{3}{2-\alpha}} \left(\frac{U}{V}\right)^{\frac{3}{2-\alpha}}.$$

Combining with expression (6) for T in terms of U/V, this leads to

(11) $$p = -\frac{1+\alpha}{3}\left(\frac{U}{V}\right).$$

The negative pressure is a consequence of the fact that –according to (9)– S decreases when V increases at constant U, i.e. at constant value of the total vortex length $\mathcal{L} = LV$, for values of α between -1 and 2. For α higher than 2 the entropy would become negative. We will comment on this point in the final paragraphs.

An increase of V implies a decrease of L, thus yielding a decrease in the total number of vortices (recall that $N \sim L^{3/2}$) and in the total entropy. Due to the relation $dW = -pdV$, with dW the work done on the system in a change of volume dV, a negative pressure means that work must be done on the system to expand it, in contrast with usual materials with positive pressure.

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3. Statistics of vortex loops

A closer understanding of the thermodynamics of counterflow vortex tangle proposed here may be achieved from a simple microscopic interpretation of the entropy in terms of the distribution function of the length of vortex loops. Let $n(l)dl$ be the number of vortex loops of length l comprised between l and $l + dl$, per unit volume. We assume that $n(l)$ has a potential form given by $n(l) \propto l^{-a}$, as [12,13]

$$n(l) = c\frac{l^{-a}}{(l_{min})^b}, \tag{12}$$

where a and c are positive dimensionless constants and l_{min} the minimum length of the vortex loops. If we want the constant c be dimensionless, we must choose $b = 4 - a$. In the following we will make this assumption. Exponent a may be linked to a possible fractal dimension of the vortex tangle [10]. Similar statistical distributions were also used in the study of cosmic strings [17]. The distribution function (12) was introduced in previous papers [12,13], but there a slightly different expression for the energy of a single loop than that used in this paper (equation (4)) were used. We will show that this new choice allow us to obtain new interesting results.

In [9], Nemirovskii obtained a potential distribution function $n(l) \propto l^{-5/2}$ for the length of the loops, as a stationary solution of a master equation describing the evolution of the length distribution function. Here, we will consider the more general expression (12).

The number N_V of vortex loops per unit volume and the line density L are [12,13]:

$$N_V = \int_{l_{min}}^{\infty} c\frac{l^{-a}}{(l_{min})^{4-a}}dl = \frac{c}{a-1}(l_{min})^{-3}, \tag{13}$$

$$L = \int_{l_{min}}^{\infty} \frac{l^{-a+1}}{(l_{min})^{4-a}}dl = \frac{c}{a-2}(l_{min})^{-2}, \tag{14}$$

where l_{min} denotes the minimum length of a vortex loop. From (14) we see that $l_{min} = (c/(a-2))^{1/2}L^{-1/2}$ which leads to:

$$< l >= \frac{a-1}{a-2}l_{min} = \frac{c^{1/2}(a-1)}{(a-2)^{3/2}}L^{-1/2}, \tag{15}$$

So, a potential distribution of the form (12) is seen to lead to relation (3) that we have taken as the starting point.

Below we will comment about the divergence of (15) for $a = 2$, which is a result of taking an infinite value for the upper limit of integration in equations (13) and (14).

As a consequence of (15) we get

$$N_V = \frac{(a-2)^{3/2}}{c^{1/2}(a-1)} L^{3/2}. \tag{16}$$

Relation (15) allows us to obtain the value of the constant c introduced in (12) as function of the constant λ. Indeed, comparing (15) with (3), we find $c = ((a-2)^3/(a-1)^2)\lambda^2$ and $N_V = (1/\lambda)L^{3/2}$. The corresponding behavior of the entropy per unit volume in terms of L and λ is [12,13]:

$$\frac{S_V}{k_B} = \frac{1}{\lambda} L^{3/2} \left[\frac{a}{a-1} + \log \frac{(a-2)\lambda^2 L^{-2}}{(a-1)^2} \right]. \tag{17}$$

Thus, a behavior in which $N_V \propto L^{3/2}$ and S with a leading term in $L^{3/2}$ may be understood from a microscopic perspective. A more detailed understanding would require a more precise calculus of the entropy density, but the presence of the logarithmic contribution in (17) is not essential, because in strict sense ϵ_V is not truly constant, but it contains a logarithmic factor in $L^{1/2}$, whose variation we have neglected (as it is usual to do), in this overview of the essential thermodynamic trends of the vortex tangle.

The energy per unit volume of the tangle, assuming $a > 2 + \alpha$, is given by:

$$U_V = \int_{l_{min}}^{\infty} n(l) U_l(l) dl = \frac{a_0 c}{a-(2+\alpha)} l_{min}^{\alpha-2} = \frac{a_0 \lambda^\alpha (a-2)^{1+\alpha}}{(a-2-\alpha)(a-1)^\alpha} L^{\frac{2-\alpha}{2}}, \tag{18}$$

where U_l, the energy of the loop of length l, is given by equation (4). The temperature of the vortex tangle is obtained as the average energy of the vortex loops:

$$k_B T = \frac{U_V}{N_V} = a_0 \frac{a-1}{a-2-\alpha} l_{min}^{1+\alpha} = a_0 \frac{(a-2)^{1+\alpha}}{(a-1)^\alpha (a-2-\alpha)} \lambda^{1+\alpha} L^{-\frac{1+\alpha}{2}}. \tag{19}$$

Results (18) and (19) are more general to that found in [12,13] (where it was found $U_V \propto L$ and $k_B T \propto L^{-1/2}$) and reduces to them if one choose $l_0 = l_{min}$ in equation (4).

This result shows that our proposal (2) for the temperature of the tangle is also corroborated from a microscopic reasoning. Furthermore, it allows

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us to find the numerical factor β introduced above equation (5) relating $< l^{1+\alpha} >$ and $< l >^{1+\alpha}$ as $< l^{1+\alpha} >= \beta < l >^{1+\alpha}$. It turns out that

$$\beta = \frac{(a-2)^{1+\alpha}}{(a-1)^{\alpha}(a-2-\alpha)} \tag{20}$$

this depends on the exponent a defining the statistics, as it was expected, as well as on the exponent α.

As a particular case, the value $a = 5/2$ obtained by Nemirovskii, leads to the following expressions for entropy, temperature and energy of the vortex tangle:

$$\frac{S_V}{k_B} = \frac{1}{\lambda} L^{3/2} \left[\frac{5}{3} + \log \frac{2\lambda^2 L^{-2}}{9} \right]. \tag{21}$$

$$k_B T = a_0 \frac{1}{3^{\alpha}(1-2\alpha)} \lambda^{1+\alpha} L^{-(1+\alpha)/2}, \tag{22}$$

$$U_V = a_0 \frac{1}{3^{\alpha}(1-2\alpha)} \lambda^{\alpha} L^{(2-\alpha)/2}. \tag{23}$$

A comment should be made on the limits of the present approach. It has been commented that $< l >$ given in (15) diverges for $a = 2$. This is a consequence of taking infinite as the upper limit of integration in (13) and (14). In fact, it would be more reasonable to take a finite value for l_{max}, related, for instance, to the diameter of the container. In this case, one gets for $a = 2$, namely, for the distribution function

$$n(l) = c \frac{l^{-2}}{l_{min}^2}, \tag{24}$$

the following results for N_V, L, U_V and S_V:

$$N_V = c(l_{min})^{-3} \left(1 - \frac{l_{min}}{l_{max}} \right), \tag{25}$$

$$L = c(l_{min})^{-2} \ln \frac{l_{max}}{l_{min}}, \tag{26}$$

$$U_V = a_0 \frac{c}{\alpha} l_{min}^{\alpha-2} \left[1 - \left(\frac{l_{min}}{l_{max}} \right)^{\alpha} \right], \tag{27}$$

$$\frac{S_V}{k_B} = \frac{c}{l_{min}^3} \left[\left(2 - \ln \frac{c}{l_{min}^4} \right) - \frac{l_{min}}{l_{max}} \left(2 - \ln \frac{c}{l_{min}^2 l_{max}^2} \right) \right]. \tag{28}$$

The connection between these quantities now depends on l_{min}/l_{max}, and it is more cumbersome than in the previous situation, although the dominant terms are still characterized by N proportional to $L^{3/2}$, U_V proportional to $L^{(2-\alpha)/2}$ and S proportional to $L^{3/2}$.

4. Analogy with black hole thermodynamics, and stability considerations

The main results of the previous analysis are

$$< l > \propto L^{-1/2}, \qquad N \propto L^{3/2}, \qquad T \propto L^{-(1+\alpha)/2}, \tag{29}$$

and S with a leading term in $L^{3/2}$. Written in terms of the tangle temperature, they are

$$N \propto T^{-3/(1+\alpha)}, \qquad U \propto T^{-(2-\alpha)/(1+\alpha)}, \qquad S \propto T^{-3/(1+\alpha)}. \tag{30}$$

However, the relation $U \propto T^{-(2-\alpha)/(1+\alpha)}$ has, apparently, some inconsistencies which seem to invalidate the definition of tangle's temperature. In fact, the decay of the vortex tangle leads to a decrease in L and, therefore, an increase in $< l >$. Thus, when the system loses energy its temperature increases. Furthermore, this behavior implies that the heat capacity of the vortex tangle is negative, because from (6) we obtain:

$$C_{loops} = \left(\frac{\partial U}{\partial T}\right)_V = -\frac{2-\alpha}{1+\alpha}\frac{A_T}{\lambda}V\left(\frac{k_B}{A_T}\right)^{-\frac{2-\alpha}{1+\alpha}} T^{-\frac{3}{1+\alpha}}. \tag{31}$$

These features share some analogies with the thermodynamics of black holes [15]. Indeed, according to the Hawking-Bekenstein theory, the entropy of a black hole is proportional to the square of the Schwarzschild radius, $S \propto R^2$, with $R = 2GM/c^2$, G, M and c being respectively the gravitational constant, the mass of the black hole and the speed of light in vacuo. Therefore, it turns out that $S(M) \propto M^2$. The energy of the black hole is $U = Mc^2$. Therefore $(\partial S/\partial U) = (1/T_{BH}) \propto M$. The thermodynamic temperature of the black hole T_{BH} satisfies the relation

$$T_{BH} \propto M^{-1} \tag{32}$$

which implies that when losing energy the temperature of the black hole increases. Further, the heat capacity of the black hole is negative: $\partial U/\partial T_{BH} \propto -M^2 < 0$. Summarizing, we have:

	VORTEX TANGLE	BLACK HOLE
Entropy	$S \propto T^{-3/(1+\alpha)}$	$S \propto M^2 \propto T_{BH}^{-2}$
Energy	$U \propto T^{-(2-\alpha)/(1+\alpha)}$	$U \propto M \propto T_{BH}^{-1}$
Heat capacity	$C \propto -T^{-3/(1+\alpha)}$	$C \propto -T_{BH}^{-2}$

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Note that for $\alpha = 1/2$, the dependence of the entropy, energy and specific heat of the tangle on the temperature is analogous to that of a black hole. At present, this seems only curiosity, but it would be of interest to compare with the results of superstring theories, where some descriptions of black holes in terms of superstrings have been proposed. Indeed, in some theories of quantum gravity, the gravitons are interpreted in terms of closed loops.

The negative character of the heat capacity implies that both vortex tangles and black holes are thermodynamically unstable. Indeed, black holes cannot exist alone, but they must radiate some electromagnetic radiation (Hawking radiation) [15], so that only the joint system constituted by the black hole and the radiation is stable. Counterflow vortex tangles are also unstable. Indeed, they cannot exist without an external heat flux. In the absence of counterflow, the total length of vortex tangle decreases due to phonon emission (sound radiation) [16].

Such a kind of process has been studied from a microscopic perspective in the analysis of a network of cosmic strings loops [17]. In fact, a negative specific heat implies a thermodynamic instability of the system [11]. This would lead the system to evolve towards a gas of loops plus a number of single long strings, comparable or longer than the horizon radius, which have a positive heat capacity. Indeed, in contrast with the gas of loops, the number of very long strings changes very slowly. This implies that in the relation $L = N_V < l >$, N_V will be fixed and $< l >$ will be proportional to L, instead to $L^{-1/2}$ as in (3). Since $U = Nk_BT$, it follows that the heat capacity is $C_{long} = N_V k_B$, which is positive. The global stability requirement is

$$\frac{1}{C_{long}} + \frac{1}{C_{loops}} \geq 0 \tag{33}$$

Therefore, a system of N long strings plus a gas of cosmic loops may be thermodynamically stable provided that the mentioned condition is realized. To get this condition, we take into account that according to (31) heat capacity of a gas of loops ($\alpha = 0$) is $C_{loops} = -2A_T^3\lambda^{-1}Vk_B^{-2}T^{-3}$. Then, the stability condition will be

$$\frac{1}{Nk_B} - \frac{k_B\lambda T^3}{2A_T^3 V} \geq 0. \tag{34}$$

This leads to the following condition for the temperature of the gas of loops in stable equilibrium with long strings

$$k_BT \leq k_BT_C = \frac{A_T}{\lambda^{1/3}}\left(\frac{2V}{N}\right)^{1/3}. \tag{35}$$

Thus, instead of imagining a pure gas of cosmic loops it seems more realistic to imagine a mixture of a few pinned lines and a gas of loops. The situation discussed in the previous paragraph is analogous to that of a Schwarzschild black hole inside a theoretical box [18]. The heat capacity of black holes is negative, and varies as M^{-2}, M being the mass of the black hole, and cannot be stable by itself, but it arrives to a stable state by emission of Hawking radiation when

$$\frac{1}{C_{blackhole}} + \frac{1}{C_{rad}} \geq 0. \tag{36}$$

5. Conclusions

In well developed counterflow superfluid turbulence we have defined an effective non equilibrium temperature, in terms of the average energy of the closed vortex loops constituting the tangle. According to this idea, the energy in the loops, in inhomogeneous systems, will go from higher T to lower T -which turns out to be, qualitatively, from lower L to higher L regions, because $k_B T_{eff}$ is expected to be of the order of $< l >\approx L^{-1/2}$. Future analyses will lead to more elaborate expressions, based on a deeper understanding of the required nonequilibrium distribution function. Furthermore, there are other relevant geometrical features of the tangle, as the orientation distribution of the local unit tangent to the vortex line or the local curvature of the vortex lines, which may also contribute to the entropy of the tangle. The meaning and diversity of temperature in non-equilibrium steady states is a current topic of research in the foundations of non-equilibrium thermodynamics and statistical mechanics [19]– [25].

Let us note, for the sake of precision and completeness, that the analysis presented here is valid for a fully developed vortex tangle, with few pinned vortices and many free vortex loops. Indeed, the thermodynamic consequences of freely moving and recombining vortex loops in fully developed turbulence is very different from that of pinned vortices. In the pinned vortices, the counterflow makes the vortex length longer, though a production of Kelvin helical waves of increasing radius, but does not change the number of vortices. Thus, for higher values of the counterflow velocity every vortex has more energy and thus there is an increase in the average length of the vortex loops and therefore an increase in the temperature as defined by (3). This behavior is analogous to that of an ideal gas, where an increase in energy does not modify the number of particles, but increases the average energy of the particles. In the case of pinned vortices in the transition from laminar to turbulence of type I, it is thus expected to follow an equipartition of energy in the several modes of Kelvin waves.

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Instead, the moving vortices continuously recombine with each other and they become smaller and more abundant when the total L becomes longer. This makes that entropy increases faster than energy in terms of L. This is precisely what happens with black holes, where the entropy S increases faster than the energy U in terms of the mass M, thus leading to a negative heat capacity, as shown in Table 1.

The analogy between the entropy of the vortex tangle and the entropy of a black hole may be of interest in the context of relations between cosmic defects -as for instance cosmic strings- and topological defects in superfluids [26].

Another aspect of possible cosmological interest is the fact that the pressure is negative, as given in Eq (11). This could be useful for thermodynamics of cosmic strings as a description of dark energy. Indeed, sufficiently negative pressure would produce an acceleration in the cosmological expansion and we have seen in (11) a simple and direct derivation of such behaviour.

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