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On some geometric and algebraic models for image analysis

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Abstract

In this paper we discuss some results of algebra and geometry and we focalize on their applications to some known models for digital image analysis. In particular, the first part of the paper is devoted to the computation of the vanishing of simplicial homology groups associated to special triangulations, which is a useful tool for obtaining information about the structure of a digital image. In the second part, some Buffon type solutions, which are of interest in the design of geophysical surveys are obtained.

Keywords: Geometric probability, Integral Geometry, image processing, simplicial homology.

AMS Subject Classification: 60D05, 52A22, 68U10, 55U10

1. Introduction.

This paper focalizes on two different applications of algebra and geometry in analysis of digital images. In the second section, we focalize on the problem of detecting information about the structure of an image, given a finite set of data. A useful tool in the solution of this problem is the computation of the simplicial homology groups of a certain triangulation associated with the finite set of data of the image. These homology groups allow us to identify the number of holes and connected components of the image. The computation of homology groups from a given triangulation is well known and the algorithm uses simple linear algebra [1]. However, the existing algorithms have run times at best cubic in the number of simplices. So the problem of computing simplicial homology is an active research area. Our approach in computing homology is theoretical: we use the combinatoric properties of the monomial ideals associated with certain simplicial complexes, in order to obtain the vanishing of the correspondent homology groups. In [2] we give a computation of the vanishing of simplicial homology

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groups for a certain class of monomial ideals. In the second section of this paper, we determine the vanishing of simplicial homology for another class of ideals.
Other questions related to digital image analysis require the knowledge of the probability of intersection of a certain spatial feature with another. A particular instance of intersection occurs when one of the features is a tile in a regular tessellation, and the other is a line, an area or a volume. Several examples follow: calculation of distance between two points referenced in the United Kingdom National Grid, determination of the number of United States Geological Survey, quadrangle map sheets required to cover a specific watershed, rasterization of a vector landcover map. In such instances, the probability that a feature intersects one of the tiles in a regular tessellation is of interest. We compute this probability using techniques coming from integral geometry. In particular, the problem of computing the probability of intersection of a geometrical object with the sides of a regular tessellation is a generalization of the classical problem, posed by Buffon in 1733, of computing the probability that a needle intersects the lines of a lattice consisting by evenly spaced parallel lines. In [3] we determine solutions of Buffon type and we apply the obtained results to the conversion of a digital image from vector to raster, in the three dimensional case. In the third section of this paper, new Buffon type solutions, which are of interest in the design of geophysical surveys, are given.

2. Simplicial homology and analysis of digital image.

In this section we examine the following problem.

Problem 2.1. *Given a finite set S of data of a digital image X, to detect some information about the structure of the image.*

One of the existing approaches to solve problems of this kind is used in [4], [5], [6] and [7] and follows a procedure which can be summarized in the following three steps:

1. modeling of X by a topological space;
2. construction of a triangulation Δ associated with S;
3. computation of the simplicial homology groups of Δ.

As we will see, the homology groups characterize the number and the type of holes and a number of connected components of X.
We recall that a simplicial complex Δ on the finite set of vertices $V = \{i_1, \dots, i_n\}$ is a collection of subsets of V such that $F \in \Delta$ whenever $F \subset G$ for some $G \in \Delta$, and such that $\{i_j\} \in \Delta$, for $j = 1, \dots, n$. The elements of Δ

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are called *faces*, and the *dimension*, $\dim F$, of a face F is the number $|F|-1$. The *dimension of the simplicial complex* Δ is $\dim\Delta = \max\{\dim F, F \in \Delta\}$.

Definition 2.1. We define the geometric realization of a simplicial complex Δ as follows: $|\Delta| = \bigcup_{F\in\Delta} |F|$.

Definition 2.2. Let X be a topological space, and $\varphi : X \to |\Delta|$ a homeomorphism. The pair (Δ, φ) is called triangulation of X.

For the reader's convenience we recall the notion of reduced simplicial homology. Let Δ be a simplicial complex with vertex set V. An *orientation on* Δ is a linear order on V. A simplicial complex together with an orientation is an *oriented* simplicial complex. Suppose Δ is an oriented simplicial complex of dimension $d-1$ and $F \in \Delta$ a face with $\dim F = i$. We write $F = [v_0, \ldots, v_i]$ if $F = \{v_0, \ldots, v_i\}$ and $v_0 < v_1 < \ldots < v_i$, and $F = []$ if $F = \emptyset$. Having introduced this notation we define the *augmented oriented chain complex of* Δ,

$$\tilde{\mathcal{C}}(\Delta) : 0 \to \mathcal{C}_{d-1} \xrightarrow{\partial} \mathcal{C}_{d-2} \to \ldots \to \mathcal{C}_0 \xrightarrow{\partial} \mathcal{C}_{-1} \to 0$$

by setting

$$\mathcal{C}_i = \bigoplus_{F\in\Delta, \dim F = i} \mathbb{Z}F \text{ and } \partial F = \sum_{j=0}^{i} (-1)^j F_j$$

for all $F \in \Delta$; here $F_j = [v_0, \ldots, \widehat{v_j}, \ldots, v_i]$ for $F = [v_0, \ldots, v_i]$. Let k be a field. We set

$$\tilde{H}_i(\Delta; k) = H_i(\tilde{\mathcal{C}}(\Delta) \otimes k), i = 1, \ldots, d-1,$$

and call $\tilde{H}_i(\Delta; k)$ the *i-th reduced simplicial homology of* Δ *with values in* k. In this definition, the empty set is considered as a face of the simplicial complex Δ with dimension -1. The *non reduced i-th simplicial homology group* is defined in a similar way but without considering the empty set as a face. Let X be a topological space with triangulation Δ. The following result is fundamental in algebraic topology [8].

Theorem 2.1.

$$\tilde{H}_i(\Delta; k) \cong \tilde{H}_i(X, k), \ \textit{for each } i.$$

Definition 2.3. We define the $i-$th Betti number as follows:

$$\beta_i := \mathrm{rank}\tilde{H}_i(\Delta; k) = \mathrm{rank}\tilde{H}_i(X, k).$$

For a topological space X, there is the following interpretation of Betti numbers: β_0 is the number of connected components minus 1, β_1 is the number of 1-dimensional holes (independent tunnel), β_2 is the number of 2-dimensional holes (enclosed voids). We approach the computation of simplicial homology groups by attaching to the simplicial complex Δ an algebraic object, the Stanley-Reisner ideal, which we are going to define.
Let $R = k[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field k. For any simplicial complex Δ we may define the ideal I_Δ of R generated by all the monomials $x_{i_1}x_{i_2}\cdots x_{i_n}$, such that $\{i_1, i_2, \ldots, i_n\} \notin \Delta$. I_Δ is called *Stanley-Reisner ideal of* Δ.
A similar definition can be given in the exterior algebra. In [2], the vanishing of simplicial homology groups for classes of simplicial complexes associated to ideals of the exterior algebra is computed. The following results enlarge the class of simplicial complexes whose vanishing simplicial homology is known.
We denote by M_q the set of all squarefree monomials of degree d in R and we introduce the lexicographic order on M_q. The lexicographic order is defined as follows: for each $u = x_{i_1}\cdots x_{i_q}, v = x_{j_1}\cdots x_{j_q} \in M_q$, $u > v$ if the first non-zero component of $(i_1 - j_1, \ldots, i_q - j_q)$ is negative.

Definition 2.4. A simplicial complex Δ is called *lexsegment* if I_Δ is a lexsegment ideal, that is, an ideal generated by a set of the form $L(u, v)$, where $L(u, v) = \{w \in M_q : v \geq w \geq u\}$.

We recall that a simplicial complex Δ is Cohen-Macaulay if for every face σ (including $\sigma = \emptyset$) and for each $i \neq \dim(\mathrm{link}_\Delta(\sigma))$ we have $\tilde{H}_i(\mathrm{link}_\Delta(\sigma; k)) = (0)$, where $\mathrm{link}_\Delta(\sigma) := \{\tau \in \Delta : \sigma \cap \tau = \emptyset, \sigma \cup \tau \in \Delta\}$. For any simplicial complex Δ which is Cohen-Macaulay the following holds.

Theorem 2.2. *(Reisner, [9]) The following conditions are equivalent:*

(a) Δ *is Cohen-Macaulay;*
(b) $\tilde{H}_i(\Delta; k) = 0$ *for all* $i < \dim\Delta$, *and the links of all vertices of* Δ *are Cohen-Macaulay.*

In the following we characterize all lexsegment simplicial complexes whose Stanley-Reisner ideal is generated in degree 2, with vanishing simplicial homology.

Corollary 2.1. *Let* $u, v \in M_2$ *and* $I_\Delta = (L(u, v))$. *Then the following statements are equivalent:*

(a) $\tilde{H}_i(\Delta; k) = 0$ *for all* $i < \dim\Delta$, *and the links of all vertices of* Δ *are Cohen-Macaulay.*

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(b) u and v have one of the following forms:
(i) $u = x_i x_{i+1}$; $v = x_{n-1}x_n$, $1 \leq i \leq n-2$;
(ii) $u = x_i x_n$ and $v \in \{x_{i+1}x_{i+2}, x_{n-2}x_{n-1}, x_{n-2}x_n\}$, for some $1 \leq i \leq n-3$;
(iii) $u = x_i x_{n-1}$, $v = x_{n-2}x_{n-1}$, $1 \leq i \leq n-2$;
(iv) $u = v = x_i x_j$, $1 \leq i < j \leq n$.

Proof. The assertion follows from Theorem 2.2 and from Theorem 1.1 in [10]. □

As a consequence, the considered class is associated to topological surfaces which have one connected component. So, for example if X is the digital image of figure 1, we may deduce that X has one connected component if the associated triangulation corresponds to a lexsegment simplicial complex satisfying condition (b) of Corollary 2.1.

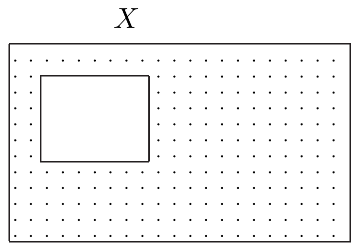

Fig. 1.

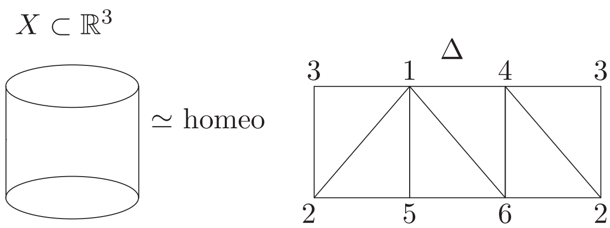

Fig. 2.

Example 2.1. Let $X \subset \mathbb{R}^3$ be the topological space consisting of a cylindric surface (see figure 2). X is homeomorphic to the triangulation Δ of figure 2. The Stanley-Reisner ideal associated with Δ is $I_\Delta =$

$(L(x_3x_5, x_4x_5)) \subset k[x_1, \ldots, x_6]$ and it satisfies condition (iii) of Corollary 2.1. Then Δ is Cohen-Macaulay. It follows from Corollary 2.1 that $\beta_0 = 0$. Then X has one connected component.

3. Buffon's needle problem and the model of a geophysical survey

A logical procedure for designing a geophysical survey when sampling an area with a regular grid can be summarized as follows:

1. modeling the expected anomaly;
2. estimate the noise level;
3. calculate the area anomaly above the noise level;
4. choose the spacing of the measurement grid.

One of the strategies used for approaching the fourth step is to leave a more complete definition of the anomaly to a later fitting. In this approach it is useful to solve the following problem.

Problem 3.1. *What is the probability of intercepting a given anomaly with a specific segment of profile and a given profile spacing?*

In [11], Luigi Sambuelli and Carlo Strobbia analyze this procedure by considering a rectangle approximating the plane projection of the anomaly shape and taking into account various ratios between the grid spacing of the rectangle sides. They determine formulas for estimating the probability of intersecting a given anomaly with a given segment of a given profile spacing. The problem of computing the probability $\mathcal{P}_s$, that a geometric object $\mathcal{T}$, intersects a segment with a fixed length s is studied in some papers of stochastic geometry [12], [13]. In this section, we compute $\mathcal{P}_s$ assuming that the object $\mathcal{T}$, approximating the anomaly area, is a circle or a pentagon. Solutions of this kind are given by A. Duma and S. Rizzo when $\mathcal{T}$ is a non regular triangle [13] and by M. Pettineo when $\mathcal{T}$ is a square or a rectangle [12].
Let a Buffon grid $\mathfrak{R}_d$, consisting of evenly spaced parallel lines with constant distance d, be given, in the Euclidian plane E_2.
In the following, the probability that a circle with constant radius r intersects a certain number k of lines of the Buffon grid, is computed.

Proposition 3.1. *If $(k-1)d \leq 2r < kd$, the probability p_k that a random circle $\mathcal{T}$, with constant radius r, uniformly distributed in a bounded region of the euclidian plane, intersects k lines of the Buffon grid $\mathfrak{R}_d$ is:*

$$p_k = \frac{2r}{kd}, k \in \mathbb{Z}, k \geq 1.$$

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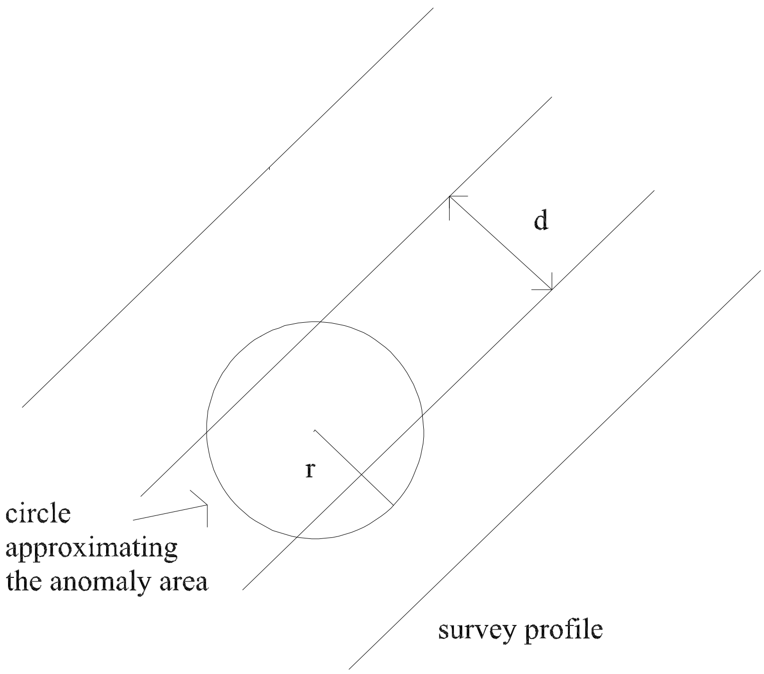

Fig. 3.

Proof. The assertion follows taking into account that the desired probability is simply given by

$$p_k = \frac{H}{L},$$

being L the distance determined by $k+1$ parallel lines, H the "effective width" of the object approximating the anomaly, that is the diameter of the circle. □

In what follows we denote by $p_{1,s}$ the probability that a circle, with constant radius r, intersects one line segment with length at least s, where s is a real number $s \leq 2r$.

Proposition 3.2. *The probability that a random circle $\mathcal{T}$, with constant radius r, uniformly distributed in a bounded region of the euclidian plane, intersects one line segment with length at least s of the Buffon grid $\mathfrak{R}_d$ is*

$$p_{1,s} = \begin{cases} \frac{1}{d}\sqrt{4r^2 - s^2}, & \textit{if } 2r \leq d; \\ \frac{1}{2kd}\sqrt{4r^2 - s^2}, & \textit{if } kd \leq 2r \leq (k+1)d, k \in \mathbb{Z}, k \geq 1. \end{cases}$$

Proof. Arguing as in the proof of Proposition 3.1, the assertion follows. □

Now, we denote by $p_{2,s}$ the probability that a circle, with constant radius r, intersects two line segments with length at least s, where s is a real number $s \leq 2r$.

Proposition 3.3. *The probability that a random circle $\mathcal{T}$, with constant radius r, uniformly distributed in a bounded region of the euclidian plane, intersects two line segment with length at least s of the Buffon grid $\Re_d$ is*

$$p_{2,s} = \begin{cases} 0, & \text{if } 2r \leq d; \\ \frac{1}{2kd}\sqrt{4r^2 - s^2}, & \text{if } kd \leq 2r \leq (k+1)d, k \in \mathbb{Z}, k \geq 1. \end{cases}$$

Proof. Arguing as in the proof of Proposition 3.1, the assertion follows. □

Corollary 3.1. *The probability that a random circle $\mathcal{T}$, with constant radius r, uniformly distributed in a bounded region of the euclidian plane, intersects a line segment with length at least s of the Buffon grid $\Re_d$ is*

$$p_s = \begin{cases} \frac{1}{d}\sqrt{4r^2 - s^2}, & \text{if } 2r \leq d; \\ \frac{1}{kd}\sqrt{4r^2 - s^2}, & \text{if } kd \leq 2r \leq (k+1)d, k \in \mathbb{Z}, k \geq 1. \end{cases}$$

Proof. The assertion follows from Proposition 3.2 and Proposition 3.3. □

Now, we consider a regular pentagon $\mathcal{P}$, with side l and vertexes A, B, C, D, E. If s is a non negative real number less than or equal to l, we obtain the probability that $\mathcal{P}$, uniformly distributed in a bounded region of the plane, intersects on $\Re_d$, with length greater than or equal to s. We consider as fundamental tile $\mathcal{F}$ a boundless strip with width d having as symmetry axle a line g of the Buffon grid. The frontier of $\mathcal{P}$ and the line g are oriented as in figure 4. We denote by ϕ the angle between g and the side AB.

Theorem 3.1. *The probability that a random pentagon $\mathcal{P}$, with constant side l, uniformly distributed in a bounded region of the euclidian plane, intersects a line segment with length at least $s \leq l$ of the Buffon grid $\Re_d$, with $d \geq \frac{4l}{\sqrt{10-2\sqrt{5}}}$ is:*

$$p_s = \frac{5l}{\pi d} - \frac{5s}{2\pi d} + \frac{s}{d}\frac{\sqrt{25 - 10\sqrt{5}}}{5}.$$

Proof. For a fixed $\phi \in [0, \pi]$, we denote by $x_s(\phi)$ the distance between two parallel chords of $\mathcal{P}$, with length s. Obviously, only the chords between

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these two chords with length s have a length greater than or equal to s. In order to compute the probability p_s we use Stoka's well known formula:

$$p_s = \frac{\int_0^\pi x_s(\phi)d\phi}{\int_0^\pi d\,d\phi}. \tag{1}$$

Since we want to consider each possible position of $\mathcal{P}$, we should take $\phi \in [0, \pi]$. But, for the existing symmetries we can consider $\phi \in \left[0, \frac{\pi}{5}\right]$. Then:

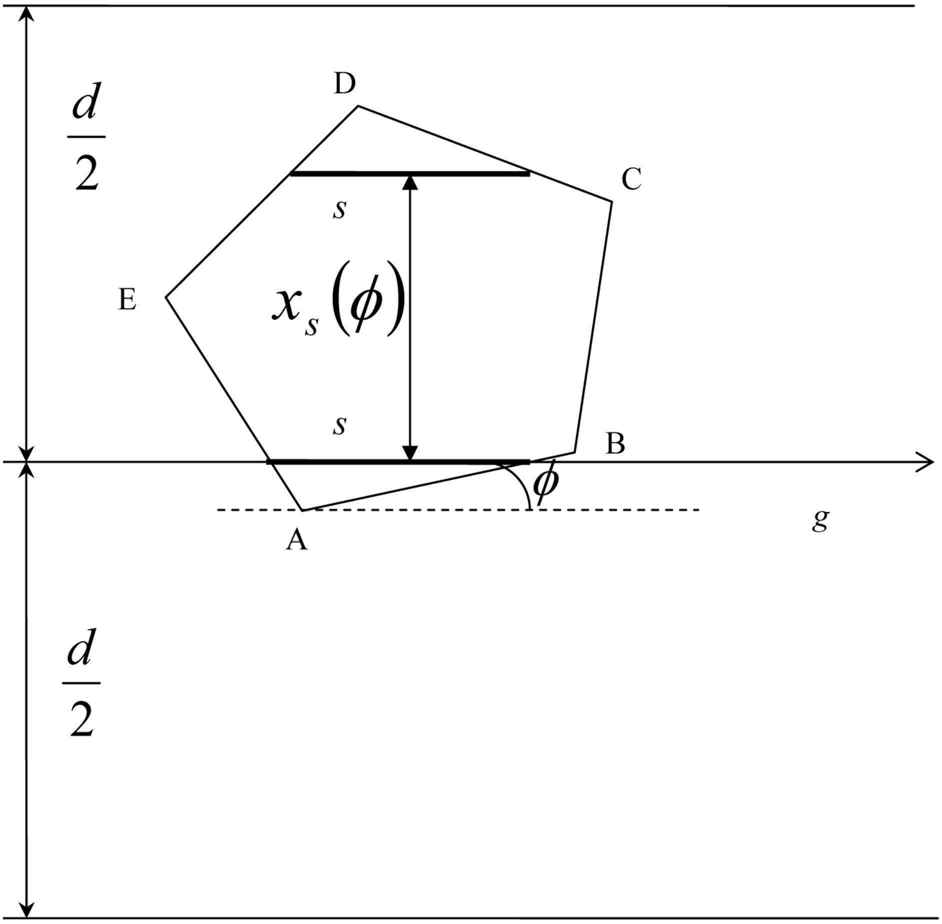

Fig. 4.

$$p_s = \frac{\int_0^{\frac{\pi}{5}} x_s(\phi)d\phi}{\int_0^{\frac{\pi}{5}} d\,d\phi} = \frac{5}{\pi d}\int_0^{\frac{\pi}{5}} x_s(\phi)d\phi. \tag{2}$$

We note that for $\phi \in \left[0, \frac{\pi}{5}\right]$ we obtain:

$$x_s(\phi) = l \sin\left(\frac{2}{5}\pi - \phi\right) + l \sin\left(\frac{\pi}{5} - \phi\right) + \\ - \frac{s \sin\left(\frac{\pi}{5} + \phi\right) \sin\left(\frac{\pi}{5} - \phi\right)}{\sin\left(\frac{3}{5}\pi\right)} + \\ - \frac{s \sin\phi \sin\left(\frac{2}{5}\pi - \phi\right)}{\sin\left(\frac{3}{5}\pi\right)}. \tag{3}$$

It follows that:

$$\int_0^{\frac{\pi}{5}} x_s(\phi) d\phi = l + s\left(\frac{\pi\sqrt{25 - 10\sqrt{5}}}{25} - \frac{1}{2}\right). \tag{4}$$

Then it follows from formulas (2) and (4), that

$$p_s = \frac{5l}{\pi d} - \frac{5s}{2\pi d} + \frac{s}{d}\frac{\sqrt{25 - 10\sqrt{5}}}{5}. \tag{5}$$ □

Finally, we determine the distribution function $\mathcal{F}$ of the chord of the pentagon, which assigns to each $s \in [0, l]$ the probability that an arbitrary chord in $\mathcal{P}$ has a chord with length less than or equal to s, that is:

$$\mathcal{F}(s) = 1 - \frac{p_s}{p_0}. \tag{6}$$

Corollary 3.2. *The distribution function $\mathcal{F}$ of the chord of the pentagon, which assigns to each $s \in [0, l]$ the probability that an arbitrary chord in $\mathcal{P}$ has a chord with length less than or equal to s, is:*

$$\mathcal{F}(s) = 1 - l + s\left(\frac{1}{2} - \frac{\pi\sqrt{25 - 10\sqrt{5}}}{25}\right). \tag{7}$$

Proof. The assertion follows from Theorem 3.1 and formula (6). □

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